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ON THE SYNTHESIS OF RESISTOR N-PORTS

by

Francis T. Boesch

Research Report PIBMRI-1143-63

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Rome Air Development Center
Research and Technology Division
Air Force Systems Command
United States Air Force
Griffiss Air Force Base, New York

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POLYTECHNIC INSTITUTE OF BROOKLYN
MICROWAVE RESEARCH INSTITUTE
ELECTROPHYSICS DEPARTMENT

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55 Johnson Street
Brooklyn 1, New York

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Francis T. Boesch

Francis T. Boesch
Instructor

Approved:

Herbert J. Carlin

Herbert J. Carlin
Head, Electrophysics Dept.

Prepared for

Rome Air Development Center
Research and Technology Division
Air Force Systems Command
United States Air Force
Griffiss Air Force Base
New York

FORWORD

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ABSTRACT

The problem of determining the necessary and sufficient conditions for a symmetric matrix to be the short-circuit admittance matrix of a transformerless resistor n-port is a classic problem in network theory. The problem is formulated topologically, and it is shown that it can be related to the well known solution for the realization of a nodal admittance matrix. In fact, the short-circuit admittance matrix and the nodal admittance matrix are shown to be related by a congruence transformation which is uniquely defined by the topology of the ports. In the special case of an n+1 terminal realization, this transformation becomes the Kron transformation. The problem of realizing a given symmetric matrix is, therefore, reduced to the determination of the configuration of port voltages. It is shown that the n+1 terminal case is identical to the problem of realiziang a given matrix as a fundamental cut-set matrix, i.e., finding a graph which has a fundamental cut-set matrix such that it is equal to the given matrix. The n+1 terminal case is studied in detail, and a simple procedure for determing the topology of the port voltages from a given matrix which contains no zero elements is derived. This synthesis procedure has the advantage of being direct, and it proceeds almost by inspection. A number of examples are given which illustrate the relative simplicity of this technique, and the synthesis of three ports is considered in detail.

Title of Report RADC-TDR-63-294

PUBLICATION REVIEW

This report has been reviewed and is approved.

Approved:

EDWARD N. MUNZER
Chief, Electronic Warfare Laboratory
Directorate of Intelligence & Electronic Warfare

Approved:

ROBERT J. QUINN, JR., Col, USAF
Director of Intelligence & Electronic Warfare

FOR THE COMMANDER:

IRVING J. GABELMAN
Director of Advanced Studies

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1.0 Introduction

The determination of the necessary and sufficient conditions that a matrix Y be the short-circuit admittance matrix of a resistor n-port without transformers is a classic problem in network theory. The purpose of this section is to serve as an introduction to the problem.

Consider an m-node network which is composed entirely of resistors, and let an n-port be defined on this network by means of attaching leads to the nodes. The maximum number of leads necessary to define the n-port is clearly 2n. If these ports are to represent independent voltages, then the minimum number is n+1. Now any node that is not utilized in defining the port voltages is superflous to the problem. If fact, these nodes may always be removed by a generalized star-to-mesh transformation which retains the positiveness of the resistors. It will, therefore, be assumed in all subsequent discussion that each of the m nodes is being used to form some port voltage. Now suppose each resistor is replaced by a directed line segment, known as an arc, whose direction is given by the direction of the current assigned to the resistor. This process yields a directed m-node graph which will be called the network graph. Notice that the network graph does not include any arcs corresponding to sources. At this point, it is convenient to utilize some of the terminology of graph theory. Some of the essential ideas of graph theory are reviewed in the next section.

1.1 Graph Theory

It will be assumed that the arc denotes the directed line segment together with its endpoints, termed nodes. These endpoints are taken as distinct, i.e., no self-arcs.

Definition 1 Arc j is said to be incident at node i if node i is an endpoint of arc j. It is incident toward node i if it is directed toward node i, and it is incident away from node i if it is directed away from i. The number of arcs incident at a node is called the degree of the node, and nodes i and j are said to be adjacent if there is an arc whose endpoints are i and j.

Definition 2 A path is a sequence of distinct arcs such that any two adjacent members of the sequence have exactly one node in common, and this common node does not belong to any other arc in the sequence. The node of the first arc which is not shared by the second arc in the sequence is called the initial node. The terminal node is defined similiarly. The path is said to be directed from the initial to the terminal node. The path is open if the initial and terminal nodes are distinct; otherwise it is

closed. A closed path is called a circuit.

Notice that this definition allows for an arbitrary direction of the arcs, i.e., the arcs may or may not agree with the direction of the path.

Definition 3 The graph is said to be connected, if there is a path between every two distinct nodes in the graph.

Definition 4 A connected graph which contains no circuits is called a tree. The arcs of a tree are called brances.

Definition 5 A graph is complete if every pair of distinct nodes are endpoints of some arc.

It is convenient to represent graphs by matrices. One such matrix is the node-arc incidence matrix.

Definition 6 The full node-arc incidence matrix S^f of a directed graph of m-nodes and ℓ-arcs is an $\ell \times m$ matrix $\left[s^f_{ij} \right]$ such that,

$$
\begin{aligned}
s^f_{ij} &= +1 \text{ if arc } i \text{ is incident away from node } j,\\
&= -1 \text{ if arc } i \text{ is incident toward node } j,\\
\text{and} \qquad &= 0 \text{ otherwise.}
\end{aligned}
$$

The reduced node-arc incidence matrix S is the matrix obtained by the deletion of any one column from S^f.

Now since every arc has exactly two endpoints, the matrix S^f has exactly one +1 and one -1 in each row. Hence S has at least one non-zero element in each row and at most two non-zero elements of opposite sign. The matrix S^f can be uniquely obtained from S by adding an extra column to S in the following manner: a zero is placed in each row that corresponds to a row in S with two non-zero elements, and a + 1 or -1 is placed in each row which corresponds to a row in S with only one -1 or +1 respectively. Any other S matrix of the same graph may be obtained from a given S martix by adding the extra column as indicated and then deleting some other column. The entire process of generating different S matrices from a given S matrix may be represented by a matrix operation with a special matrix.

Definition 7 A Shekel matrix T_k of an m node graph is a square $(m-1) \times (m-1)$ matrix which is diagonal except for the k^{th} column whose elements are all -1. All the diagonal elements but the k^{th} are +1.

Now let S_o and S_1 be two distinct reduced incidence matrices of the same graph*. By a straight forward application of the definition of matrix multiplication,

*The terms "reduced incidence matrix" and "incidence matrix" will be used to denote the reduced node-arc incidence matrix and the full node-arc incidence matrix respectively.

the reader may easily verify that

$$S_1 = S_o T_k. \tag{1-1}$$

In fact, S_1 corresponds to cancelling the k^{th} column of S_o and replacing it by the column of S^f that was deleted in forming S_o.

There is a unique correspondence between a graph and its incidence matrix. Certainly, the incidence matrix is uniquely defined for any graph; alternatively, the graph is uniquely defined by its incidence matrix. The validity of the latter statement is easily demonstrated by considering any rectangular $\ell \times m$ matrix Q whose elements are +1, -1, and 0. If each row of Q contains exactly one +1 and one -1 and the remaining elements are zeros then a graph may be constructed whose incidence matrix is Q as follows: form m nodes and connect nodes i and j by arc k if the q_{ki} and q_{kj} elements are non-zero. Arc k is directed away from node i if $q_{ki} = +1$; otherwise it is directed toward node i.

1.2 The Nodal Admittance Matrix

It has been assumed that currents are defined for each resistor in obtaining the network graph. Now let the resistor voltages be defined as voltage drops, i.e., the current is directed from the positive node to the negative node. Let one node be chosen as a reference, and define node potentials as positive node voltages with respect to this reference. The column vector $\underset{\sim}{v}$ will denote all the arc voltages, and the column vector $\underset{\sim}{P}$ will denote the node potentials. Furthermore, let S be the reduced incidence matrix of the network graph which corresponds to the deletion of the reference node column. Thus, by Kirchhoff's voltage law,

$$\underset{\sim}{v} = S\underset{\sim}{P} \tag{1-2}$$

Now let an n-port be described on the network by means of attaching leads to the nodes; it is assumed that n independent voltages are defined between pairs of leads. A new graph is then obtained by adding arcs to the network graph in the following manner: an arc is placed between two nodes if and only if a port voltage is defined between these nodes, and the arc is directed from the positive to the negative polarity. This graph will be referred to as the augmented graph. It is important to note that the arcs corresponding to the ports are directed opposite to the usual convention for port currents. Let $\hat{S}$ be the reduced incidence matrix of the augmented graph with respect to the same reference node, and let the last n rows of $\hat{S}$ correspond to those arcs which represent the port voltages. Therefore, S may be written as

$$\hat{S} = \left[\frac{S}{A}\right] \tag{1-3}$$

where A is the reduced incidence matrix of the subgraph consisting of the port voltages, and S is the reduced incidence matrix of the network graph. Since A designates the connection of the ports, it will be referred to as the connection matrix.

If $\underset{\sim}{V}$ is the column vector of the port voltages, then the column vector $\underset{\sim}{\pi}$ of the arc voltages of the augmented graph is given by:

$$\underset{\sim}{\pi} = \left[\begin{array}{c} \underset{\sim}{v} \\ \hline \underset{\sim}{V} \end{array}\right] \tag{1-4}$$

Since there are no nodes being added in forming the augmented graph, the vector $\underset{\sim}{P}$ remains unchanged. Applying Kirchhoff's voltage law to the augmented graph:

$$\underset{\sim}{\pi} = \hat{S}\underset{\sim}{P} = \left[\begin{array}{c} \underset{\sim}{v} \\ \hline \underset{\sim}{V} \end{array}\right] = \left[\begin{array}{c} S \\ \hline A \end{array}\right] \underset{\sim}{P} \tag{1-5}$$

If $\underset{\sim}{I}$ is the column vector of port currents, then the column vector $\underset{\sim}{\phi}$ of the arc currents of the augmented graph is given by

$$\underset{\sim}{\phi} = \left[\begin{array}{c} \underset{\sim}{i} \\ \hline -\underset{\sim}{I} \end{array}\right] \tag{1-6}$$

Kirchhoff's current law applied to the augmented graph yields

$$0 = \left[S' \,|\, A' \right] \underset{\sim}{\phi} = \left[S' \,|\, A' \right] \left[\begin{array}{c} \underset{\sim}{i} \\ \hline -\underset{\sim}{I} \end{array}\right] \tag{1-7}$$

Thus,

$$S'\underset{\sim}{i} = A'\underset{\sim}{I} \tag{1-8}$$

From the definition of the A matrix, it follows that the elements of the vector $A'\underset{\sim}{I}$ are the total applied currents entering the nodes; these currents are referred to as the nodal currents and denoted by the vector $\underset{\sim}{J}$, i.e.,

$$\underset{\sim}{J} = A'\underset{\sim}{I} \tag{1-9}$$

Now let $\hat{G}$ be a diagonal matrix of arc conductances such that

$$\underset{\sim}{i} = \hat{G}\underset{\sim}{v} \tag{1-10}$$

Hence,

$$\underset{\sim}{J} = A'\underset{\sim}{I} = S'\underset{\sim}{i} = S'\hat{G}\underset{\sim}{v} = S'\hat{G}S\underset{\sim}{P} \tag{1.11}$$

The matrix $S'\hat{G}S$ is called the nodal-admittance matrix and will be denoted by G, and the matrix equation relating $\underset{\sim}{J}$ to $\underset{\sim}{P}$ is called the nodal admittance equations. It can be shown that the nodal-admittance equations yield the most general solution to the network. If the full node arc incidence matrix is used in this development, then the

matrix $S^{f'}\hat{G}S^{f}$ is the indefinite admittance matrix. Let

$$G = \left[g_{ij}\right] \tag{1-12}$$

From the definition of S and G it follows that:

g_{ii} is the sum of the conductances connected to node i.

and

g_{ij} is the negative of the conductance between nodes i and j.

Thus the nodal-admittance matrix has the property that the diagonal elements are not less than the sum of the off diagonal terms. The nodal-admittance matrix may be uniquely chatacterized by this property. It is convenient to introduce some terms.

Definition 8 A constant matrix Y is said to be dominant if all the diagonal terms are not less than the sum of the magnitudes of all other elements in the same row, i.e.,

$$y_{ii} \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} |y_{ij}|$$

Definition 9 A matrix is hyperdominant if and only if it is dominant, and in addition all the off-diagonal terms are nonpositive.

The properties of the nodal admittance matrix are contained in the following theorem.

Theorem 1 The necessary and sufficient conditions that a real, constant, symmetric nxn matrix G be the nodal admittance matrix of a network composed entirely of positive resistors with the node potentials defined as positive with respect to the reference node is that G be hyperdominant.

Proof The necessity follows immediately from the previous observations on the elements of G. The sufficiency follows from a synthesis of the network. The conductance between the i^{th} and j^{th} nodes is taken as $|g_{ij}|$, and the conductance between the i^{th} node and the reference is given by

$$\sum_{j=1}^{n} g_{ij}.$$

Since G is hyperdominant, all the conductances are non-negative. From the properties of the nodal-admittance matrix, it is clear that the given G is the nodal-admittance matrix of the constructed network.

The short-circuit admittance matrix is related to the nodal-admittance matrix by a simple transformation, which will be developed in the next section.

1.3 The Kron Transformation

In the previous section, it is shown that the port voltages and currents are related to the nodal voltages and currents by the following transformations:

$$\underset{\sim}{J} = A' \underset{\sim}{I} \tag{1-13}$$

and

$$\underset{\sim}{V} = A \underset{\sim}{P} \tag{1-14}$$

It is interesting to note that the total power absorbed by the network is invariant under this transformation, viz,

$$\underset{\sim}{V}'\underset{\sim}{I} = \underset{\sim}{P}'A'\underset{\sim}{I} = \underset{\sim}{P}'\underset{\sim}{J} \tag{1-15}$$

Now if Y denotes the short-circuit admittance matrix of the network, then

$$\underset{\sim}{I} = Y\underset{\sim}{V} = YA\underset{\sim}{P} \tag{1-16}$$

and

$$\underset{\sim}{J} = A'\underset{\sim}{I} = A'YA\underset{\sim}{P} \tag{1-17}$$

Recall that the connection matrix A which is the incidence matrix of the ports has order n x (m-1), where

$$n \leq m - 1. \tag{1-18}$$

Now if the ports are to represent independent voltages then the subgraph of the augmented graph defined by A does not contain any circuits. It is shown in a subsequent section that this implies that the rank of A is n.

Although the matrix A'YA is a transformation of node potentials into node currents, it can not generally be identified as the nodal-admittance matrix. The reason being that the rank of A'YA can not exceed n, but the rank of G may be n-1. However, when

$$m = n + 1, \tag{1-19}$$

the connection matrix is square and non-singular. Thus existence of Y implies that any $\underset{\sim}{V}$ is admissible, and the equation

$$\underset{\sim}{V} = A\underset{\sim}{P} \tag{1-20}$$

implies that any $\underset{\sim}{P}$ vector is allowable. Hence, in the n + 1 node case,

$$\underset{\sim}{J} = A'YA\underset{\sim}{P} = G\underset{\sim}{P} \tag{1-21}$$

for all $\underset{\sim}{P}$; therefore,

$$G = A'YA. \tag{1-22}$$

This congruence transformation is known as the Kron transformation.

A transformation may also be obtained in the general case if it is assumed that G and Y are non-singular. Hence,

$$\underset{\sim}{V} = A\underset{\sim}{P} = AG^{-1}\underset{\sim}{J} = AG^{-1}A'\underset{\sim}{I} \tag{1-23}$$

and

$$\underset{\sim}{V} = Y^{-1}\underset{\sim}{I} \tag{1-24}$$

The existence of Y^{-1} implies that any $\underset{\sim}{I}$ is allowable; hence

$$Y^{-1} = AG^{-1}A' \tag{1-25}$$

The Kron transformation enables the problem of realizing an arbitrary matrix Y as the short-circuit admittance matrix of an n-port defined on n+1 nodes to be related to the realization of a nodal-admittance matrix.

Theorem 2 The necessary and sufficient condition that a real, constant, symmetric matrix Y be the short-circuit admittance martix of an n-port defined on an n+1 node resistor network is that there exists a connection matrix A such that A'YA is hyperdominant.

Proof The necessity follows from the previous discussion; the matrix A'YA is the nodal-admittance matrix and, hence, hyperdominant. The sufficiency follows from the synthesis of A'YA as a nodal-admittance matrix G; see theorem 1. It follows from the non-singularity of A that the short-circuit admittance of the network with nodal-admittance matrix G is Y when the ports are defined in accordance with the connection matrix, viz.,

$$A'YA = G \tag{1-26}$$

and

$$Y = (A^{-1})'G\,(A^{-1}) \tag{1-27}$$

The realization of a short circuit admittance matrix on n+1 terminals is identical to the problem of realizing a fundamental cut-set matrix.

1.4 The Cut-Set Matrix

From the previous section, it is evident that

$$G = A'YA = S'GS \tag{1-28}$$

for an n-port defined on n+1 terminals. Since A is non-singular, it has an inverse which will be denoted by B. Hence

$$Y = B'S'\,GSB \tag{1-29}$$

It is shown in a following section that the connection matrix corresponds to a tree in the n+1 terminal case. The matrix SB is thus defined as the fundamental cut-set matrix C of the graph which corresponds to the tree defined by A. This definition

of the cut-set matrix is not the classical definition, and it is not obvious that they are equivalent. (22) Since this is not relevant to the discussion, the proof of the equivalence is relegated to Appendix I. The problem of realizing a cut-set matrix is a classic problem in graph theory. Clearly this problem is related to the resistor synthesis problem. Suppose C is the fundamental cut-set matrix of a graph with respect to some tree; certainly the matrix $C'C$ is the short-circuit admittance matrix of a network of unit conductances. It is shown in Appendix II that if $C'C$ is a realizable short-circuit admittance matrix then C is a cut-set matrix.

2.0 The Connection Matrix

Recall that the connection matrix is an incidence matrix of a graph without circuits. In general, a graph without circuits is clearly a disjoint collection of trees. It is important, therefore, to study the properties of the incidence matrix of a tree. The following lemmas provide for a simple proof of the properties of the connection matrix.

Lemma 1 Every tree has at least two nodes of degree one.

Proof Consider the collection of all paths in a tree. Certainly there is at least one path with the property that no other path in the tree contains a greater number of branches. Now the initial and terminal nodes of this path must be distinct, and they must have degree one otherwise this would not be the path with the longest number of branches.

Lemma 2 If a tree has n nodes, then it has n-1 branches.

Proof The lemma is obvious for a tree with 2 nodes. Now consider any tree with n + 1 nodes, and remove the branch which is incident at a node of degree one. The resulting configuration is certainly a tree since it is connected and contains no circuits. However, this yields an n node tree, and the induction hypothesis implies that the n node tree has n-1 branches. Thus, since the n+1 node tree differs from the n node tree by exactly one branch, the n +1 node tree has n-1+1 branches.

An obvious consequence of lemma 2 is that the reduced incidence matrix of a tree is a square matrix.

Lemma 3 The reduced incidence matrix of a tree is non-singular.

Proof For a tree of 2 nodes, it is clear that the 1 x 1 reduced incidence matrix is non-singular. Now consider any n+1 node tree incidence matrix. Since there is at least one node of degree one, there is at least one row which contains only one non-zero element. Thus, the determinant of the n + 1 node incidence matrix vanishes if and only if the minor defined by the single non-zero element vanishes. However, this

minor is the incidence matrix for the n node tree obtained by removing the branch in the n+1 node tree which is incident at the node of degree one. Again the proof is completed by induction.

The proof of the next lemma is well known; however, it is included for the sake of completeness.

Lemma 4 Every connected graph contains a tree which includes all the nodes of the graph.

Proof The proof follows by actually constructing a tree. Consider any arc in a connected graph. If this arc is included in some circuit, then the removal of it does not render the graph disconnected. If this arc is not contained in any circuit, then the removal of it renders the graph disconnected. The tree is constructed by considering any arc of the graph. If it is included in some circuit remove it; this yields a connected graph with at least one less circuit. By continuing in this fashion, all the circuits may be destroyed without losing the connected property. Thus, the process finally terminates when a tree is obtained.

Theorem 3 The rank of the reduced incidence matrix of any connected graph of n nodes is $n-1$.

Proof By lemma 4, it follows that the incidence matrix contains a minor of order $(n-1) \times (n-1)$ which is the incidence matrix of a tree. By lemma 3, this minor does not vanish; hence, the rank is $n-1$.

These results can be used to determine the properties of the connection matrix. The fact that the rows of the connection matrix are linearly independent will follow immediately from the following theorem.

Theorem 4 A set of rows of an incidence matrix are linearly independent if and only if they define a subgraph without circuits.

Proof First observe that in a circuit every node is of degree two. Thus each column has exactly two non-zero elements. Hence, after an appropriate sign change, the sum of the rows is zero. Therefore, linearly independent rows do not correspond to a circuit. To prove the converse, consider a graph without circuits. Clearly, the graph is a disjoint collection of trees. Consider a node in any one of these trees, and add branches connecting this node to each other tree such that only one branches connects the chosen tree and any other tree. The resulting configuration is a tree since it is connected, and no circuits were created by this process. Thus the original rows are some subset of the rows of a tree incidence matrix. Since the tree incidence matrix is non-singular, these rows are linearly independent.

As stated in a previous section, the rows of the connection matrix are linearly independent. It will now be shown that in the case of an n-port defined on n+1 nodes the connection matrix is the incidence matrix of a tree. Now there are no closed circuits; it remains to be shown that it is connected.

Theorem 5 In a graph with no circuits, if the number of branches is one less than the number of nodes, then the graph is a tree.

Proof Since there are no circuits it must be a disjoint collection containing r distinct trees. Now for each tree, if n_i denotes the number of nodes and b_i the number of branches, then $b_i = n_i - 1$. Thus, if b is the total number of branches and n the total number of nodes, i.e.,

$$n = \sum_{i=1}^{r} n_i \tag{2-1}$$

$$b = \sum_{i=1}^{r} b_i \tag{2-2}$$

then

$$b = n - r \tag{2-3}$$

However, it is assumed that,

$$b = n - 1 \tag{2-4}$$

Thus the number of disjoint trees is one; the graph is a tree.

2.1 A Standard Form for a Square Connection Matrix

Recall that when the connection matrix is square it corresponds to a tree. Thus there is at least one node of degree one, or alternatively there is at least one row of the connection matrix with only one non-zero element. Consider all the rows of the connection matrix A which have only one non-zero element. Since A is non-singular, no two of these non-zero elements can occupy the same column. Now by column and row permutations, which correspond to renumbering the branches and the nodes, these non-zero elements can be arranged to form a diagonal principal minor in the upper left-hand corner. Now consider the principal submatrix obtained by cancelling these rows of single non-zero elements. The determinant of this submatrix is equal in magnitude to the determinant of A. The rows of this submatrix contain at most two non-zero elements of opposite sign. If there is no row with exactly one non-zero element, then the sum of the columns of this submatrix would be zero; thus it would be singular. Hence, there is at least one row with only one non-zero element. Let all the rows with only one non-zero element be permuted to the upper left-hand corner of the submatrix. If this process is continued, then a lower triangular matrix is obtained by a permutation of rows and columns. Now by reversing the signs of

appropriate rows, the diagonal elements may be made positive.

Further properties of this standard form may be obtained by partitioning the lower triangular matrix corresponding to the rows of single non-zero entries, i.e., so that the diagonal partitions are all unit matrixes. This partition is shown in Figure 1.

$$\begin{bmatrix} 1_{r_1} & & & & & & \\ A_{21} & 1_{r_2} & & & & & \\ & A_{32} & 1_{r_3} & & & & \\ & & A_{43} & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \\ & & & & & & 1_{r_k} \end{bmatrix}$$

Fig. 1 The Standard Form of A

The only non-zero submatrices are $A_{i+1,\ i}$. Indeed suppose that there is a non-zero element elsewhere. Then all other elements in this row are zero except the diagonal element. However, this implies that this row belongs to a different partition.

A further ordering may be obtained by permutations of the off-diagonal submatrices. Since any row of the connection matrix contains at most two non-zero elements, the submatrices $A_{i+1,\,i}$ consist of non-overlapping columns of -1 elements, which, could be arranged in order of decreasing length without disturbing the standard form. However, this possibility is immaterial to further discussions, and it will be disregarded. It is important to note that this standard form is not canonic in the sense that two isomorphic trees may have different standard forms. In this context, two trees are said to be isomorphic if they differ only in the numbering of ports and nodes, i.e., their full incidence matrices are permutations of each other.

It is significant to observe the relation of the standard form to the Kron transformation. Consider two connection matrices which differ in a numbering of the nodes, $A_1 = A_oP_1$, where P_1 is an elementary matrix obtained by some column permutation of the unit matrix. Certainly $A_1' YA$, is hyperdominant if and only if A_o' YA_o is hyperdominant. Now if the port voltages are renumbered and reoriented via $A_3 = P_2A_o$, where P_2 is an elementary matrix obtained by some row permutation and sign change on a unit matrix, then the permutation $Y_3 = P_2Y_oP_2'$ is realizable via

$A_3' Y_3 A_3$ if $A_o' Y_o A_o$ is hyperdominant. The validity of this statement follows immediately from the observation that P_2 is orthogonal.

The standard form of the connection matrix has an immediate application in the study of the square submatrices of an arbitrary incidence matrix.

Definition 10 A rectangular matrix all of whose elements and subdeterminants are +1, -1, or 0 is called totally unimodular. The term totally unimodular should not be confused with the term unimodular which is usually reserved for square matrices whose determinant is +1, -1, or 0. Many authors use the term E-unimodular to denote totally unimodular.[11, 22]

Theorem 6 The full incidence matrix of any graph is totally unimodular.

Proof Consider any square submatrix. The columns of this submatrix are either linearly dependent or linearly independent. If they are linearly dependent, then the determinant of this submatrix is zero. If they are linearly independent, then by theorems 3 and 4 this submatrix corresponds to a tree, viz., it is the reduced incidence matrix of some tree. Now from the standard form of the tree incidence matrix, it is clear that the determinant of the reduced incidence matrix of a tree is either +1 or -1. Thus the full incidence matrix is totally unimodular. Therefore, any reduced incidence matrix is also totally unimodular. It is also clear that the standard form of a tree incidence matrix is totally unimodular.

Examples of the standard form of the connection matrix will be given after the topological interpretation is discussed. Now the inverse of the connection matrix and the corresponding standard form will be discussed.

2.2 The Inverse of a Square Connection Matrix

The inverse of the connection matrix A will be denoted by B. From the totally unimodular property, it is obvious that the elements of B are +1, -1, or 0. Now suppose A is in standard form as described in the previous section. Let B be partitioned conformally with A. This is represented in Figure 2; clearly B is lower triangular and has unit elements on the diagonal.

$$
\begin{bmatrix}
1_{k_1} & 0 & 0 & 0 & \vdots & \vdots \\
B_{21} & 1_{k_2} & 0 & 0 & \vdots & \vdots \\
B_{31} & B_{32} & \ddots & \vdots & 0 & \vdots \\
\vdots & \vdots & \vdots & 1_{k_j} & \vdots & \vdots \\
B_{j1} & B_{j2} & \vdots & \vdots & \ddots & \vdots \\
B_{r_1} & B_{r_2} & \vdots & \vdots & \vdots & 1_{1_r}
\end{bmatrix}
\begin{bmatrix}
1_{k_1} & 0 & 0 & \vdots & \vdots & \vdots \\
A_1 & 1_{k_2} & 0 & \vdots & \vdots & \vdots \\
0 & A_2 & \ddots & 0 & \vdots & 0 \\
0 & \vdots & A_{i-1} & 1_{k_i} & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & A_{r-1} & 1_{k_r}
\end{bmatrix}
= 1_n
$$

Fig. 2 A and B partitioned conformally

The submatrices of B can be related to those of A in a simple manner.

Theorem 7 Let A be in standard form and partition B conformally with A. Then

i. $B_{k\ell} = - B_{k\,(\ell+1)} A_\ell$ for $\ell \leq k-1$

ii. $B_{k\ell} = (-1)^{k-\ell} A_{K-1} A_{K-2} \cdots A_\ell$ for $\ell \leq k-1$

iii. $B_{\ell(\ell-1)} = - A_{(\ell-1)}$ for $\ell \leq k-1$

Proof i. Multiply the j^{th} row partition of B into the ℓ^{th} column partition of A. Thus

$$B_{j\ell} = -B_{j(\ell+1)} A_\ell \tag{2-5}$$

ii. This result is easily obtained by an iteration of the above recursion relation.
iii. This result follows immediately from the second by allowing k= ℓ-1.

There are many other properties of B which may be obtained from the standard form. For example, it can be shown that the elements of the inverse of the standard form of A are non-negative. This proof will be reserved, however, for the topological interpretation. It may also be shown that the inverse of any connection matrix is totally unimodular; however, this result is not related to the problem under consideration.

2.3 Topological Interpretation of the Connection Matrix

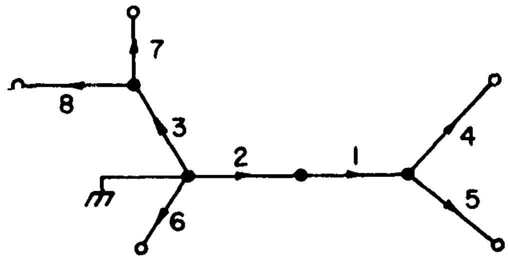
The standard form of the connection matrix and the inverse of the connection matrix have a simple topological interpretation which is of utility in deriving their properties. It is convenient to define a few terms.

Definition 11 A tree in which all the branches are incident at a common node is said to be a star.

Definition 12 Let i and j be two nodes in a graph. The graph obtained by allowing nodes i and j to coalesce is called the graph obtained by shorting nodes i and j.

If nodes i and j are connected, then the graph obtained by shorting nodes i and j will contain a new closed path. Furthermore, if i and j are adjacent, then the graph obtained by shorting i and j will contain a self-arc. If this self arc is removed, then the resulting graph is said to be the graph obtained by shorting the arc which is incident at nodes i and j.

Consider the standard form of A as shown in Figure 1. The first r_1 rows correspond to all the branches which are incident at the reference node. Thus they form a sub-tree which is a star at the reference node. Now consider the graph obtained by shorting all the branches which are incident at the reference node. The branches which are incident at the reference node in the new graph correspond to rows $r_1 + 1$ through r_2 in the standard form. These topological operations simplify the procedure for obtaining the standard form of the connection matrix. Indeed, it proceeds by inspection. Consider the tree shown in Figure 3.



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Fig. 3 A Tree and a Standard Form of the Connection Matrix

When the reference node is taken as indicated in the figure, branches 2, 3, and 6 form the first partition of the standard form. When these branches are shorted, it follows that branches 1, 7, and 8 correspond to thenext partition. Finally, when these branches are also shorted, the last partition corresponds to branches 4 and 5. The non-zero off-diagonal submatrices can be obtained from the incidence relations. In the example, branches 7 and 8 are taken as incident 4 and 5 are taken as incident at the third node of the second star. Thus the permutation of branches which yields the standard form shown in Figure 3 is

$$12345678 \longrightarrow 36278145 \qquad (2\text{-}6)$$

The inverse of a connection matrix may also be given a topological interpretation by utilizing the electrical properties. Recall that

$$\underset{\sim}{V} = A \underset{\sim}{P}$$

thus

$$\underset{\sim}{P} = B \underset{\sim}{V}$$

Let p_i denote the element in the i^{th} row of $\underset{\sim}{P}$ and let V_j denote the element in the j^{th} row of $\underset{\sim}{V}$. Clearly,

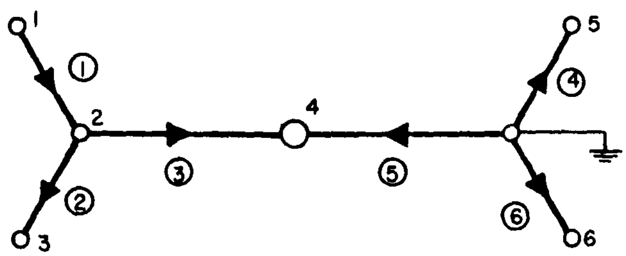
$$b_{ij} = \frac{b_i}{V_j} \quad \text{when} \quad V_k = 0 \text{ for all } k \text{ except } k = j$$

Now consider the tree path from node i to the reference; this path is unique since there are no circuits in a tree. If V_j is not contained in this path, then when $V_k = 0$ for all k but k=j node i is shorted to the reference. Hence $p_i = 0$, and $b_{ij} = 0$. Now assume that this path is directed from node i to the reference. If V_j is included in this path then it is included in the same direction, i.e., the positive sense, or it is included in the opposite direction. Thus for $V_k = 0$ for all k but k=j, $p_i = V_j$ if V_j is included in the path in the positive sense. Likewise $p_i = -V_j$ in the opposite sense. This is summarized as follows:

$$\begin{aligned} b_{kj} &= +1 \text{ if the tree path from node k to ground} \\ &\qquad \text{includes port j in the positive sense.} \\ &= -1 \text{ if the tree path from node k to ground} \\ &\qquad \text{includes port j in the negative sense.} \\ &= 0 \text{ otherwise.} \end{aligned}$$

An important application of the topological interpretation of B is the inversion of A matrix. If the graph is drawn and the topological definition of B is utilized, then the matrix may be inverted almost by inspection. An A matrix and its corresponding tree are shown in Figure 4. The elements of B are determined as follows: the path

from node 1 to the reference only includes branches 1 , 3 , and 5 . Thus the only non-zero terms in the first row of B are b_{11}, b_{13}, and b_{15}. Now 1 and 3 are included in the positive sense and 5 in the negative sense. Hence $b_{11} = 1$, $b_{13} = 1$, and $b_{15} = -1$. The entire B matrix is obtained by proceeding in this fashion; it is shown in Figure 4.



ports are given by the encircled numbers.

$$A = \begin{bmatrix} +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & +1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Fig. 4 A tree and its A and B matrices

The topological interpretation of B may also be utilized in obtaining properties of the B matrix. An example is the following theorem.

Theorem 8 All the nonzero elements in any column of B have the same sign.

Proof Suppose elements b_{rp} and b_{kp} had opposite signs. Then the ground paths from nodes k and r would include p in opposite directions. Thus the nodes at which p is incident occur in opposite order on the two paths through them. This implies that these two nodes have distinct ground paths. This is impossible since it requires a closed path in the tree.

This theorem leads to the conclusion that there exists a diagonal matrix D such that BD contains no negative elements. Recall that this operation corresponds to reorienting the ports so that the nonzero elements of B D are all +1. Hence any ground path only includes ports in the positive sense of the path. When a tree has this property, it is said that all ports are directed toward ground.

3.0 N+1 Node Resistor N-Ports

The properties of a square connection matrix and the Kron transformation enable a detailed analysis of the N+1 node case. First a summary of the contributions to the resistor problem is given, and then the N+1 node case is studied in detail. A simple synthesis procedure is given for this case.

3.1 Summary of Resistor N-Port Problem

An excellent survey of the history of the resistor n-port problem is given in a paper by Weinberg and Slepian[23]. A brief survey of the contributions to the problem are presented. First it is necessary to define a term which is used in the study of resistor n-ports.

Definition 13 A constant matrix is said to be paramount if every principal minor is not less than the magnitude of any other minor formed with the same rows.

For the sake of clarity, it should be observed that the term short-circuit admittance matrix will be reserved for the n-port description of a network, and the term nodal-admittance matrix will be used to denote the matrix which transforms a set of positive node potentials into the set of current sources that enter the nodes.

It is well known that a positive definite, symmetric, constant matrix may always be realized as the short-circuit admittance matrix of an n-port of resistors and transformers[8]. Now the conditions that the network does not contain transformers is essentially a topological constraint; the problem then amounts to characterizing this constraint. Cederbaum has shown that paramountcy is a necessary condition based on the nonnegativeness of the resistors and the no-gain property of transformerless networks[9, 10]. Probably the first sufficient condition was given by Foster who showed that dominance is sufficient for the realization of a nonsingular symmetric matrix Y as the short-circuit admittance matrix of an n-port defined on 2n terminals[23]. Tellegan proved that paramountcy was sufficient for the realization of a symmetric 3x3 impedance or admittance matrix on five terminals[24]. Cederbaum has recently shown that paramountcy is not sufficient in the general case of $n > 3$ and any number of terminals[13]. Subsequently, the majority of effort has been directed toward the realization of an n+1 terminal graph.

There are tow techniques available in the (n+1) terminal case. The first is an algorithm due to Cederbaum which is based on the observation that Y is realizable as a short-circuit admittance matrix if and only if

$$Y = C^t \Delta C, \tag{3-1}$$

where Δ, the branch conductance matrix, is a diagonal matrix of positive constants, and C is the cut-set matrix[12]. Cederbaum gives a procedure by which an arbitrary

matrix Y may be factored into the indicated congruence transformation where C is totally unimodular[9, 22]. Having performed the factorization, C must be tested by the Gould algorithm to determine if it is a cut-set matrix[15].

An alternative approach, based on concepts initiated by Guillemin, was given by Biorci and Civalleri who showed that the configuration of port voltages could be determined from the signs of the off-diagonal terms of Y[2, 3, 4, 5, 14]. With a knowledge of the port voltage configuration, the synthesis is immediate via a Kron transformation.

The relation between the short-circuit admittance matrix and the corresponding connection matrix is discussed in detail in the next section. A synthesis procedure is then given which has the advantage of eliminating the Gould algorithm used by Cederbaum and of avoiding the "tree growing" process utilized by Biorci and Civalleri.

3.2 Properties of the Short-Circuit Admittance Matrix

The signs of the elements of the short-circuit admittance matrix have a unique relation to the elements of the connection matrix. These properties are studied by utilizing the Kron transformation and the topological interpretation of the connection matrix. In order to do so, it is convenient to define the sign of a matrix.

Definition 14 The sign of a matrix Y of nonzero constants is an n x n matrix denoted by

$$\operatorname{sgn}(Y) = \left[s_{ij} \right] = \left[\operatorname{sgn} y_{ij} \right]$$

where

$$s_{ij} = +1 \quad \text{if } y_{ij} > 0$$

and

$$s_{ij} = -1 \quad \text{if } y_{ij} < 0$$

Theorem 9 Let H be a hyperdominant matrix of nonzero elements. If all ports are directed toward ground, then sgn (HB) = 2B - U, where U is a square matrix whose elements are all +1.

Proof Consider

$$s_{ij} = \operatorname{sgn} \left[\sum_{K=1}^{N} h_{ik} b_{kj} \right] = \operatorname{sgn} \left[b_{ij} h_{ii} - \sum_{\substack{K=1 \\ K \neq i}}^{N} |h_{ik}| b_{kj} \right] \tag{3-2}$$

Now all the elements of B are nonnegative, and H is hyperdominant. Thus s_{ij} is completely determined by b_{ij}. Suppose $b_{ij} = 0$; then $s_{ij} = -1$. If $b_{ij} = +1$, then $s_{ij} = +1$. Hence

$$s_{ij} = 2b_{ij} - 1 \tag{3-3}$$

Corollary 1 Let B correspond to an arbitrary direction of ports, and let BD be the reorientation which directs all ports toward ground. If H is a hyperdominant matrix of nonzero elements, then sgn (HB) = 2B - UD.

Proof Certainly,

$$\operatorname{sgn}(HB_o) = 2B_o - U \tag{3-4}$$

where

$$B_o = BD \tag{3-5}$$

Thus,

$$\operatorname{sgn}(HB_o)\,D = 2B_oD - UD = 2B - UD \tag{3-6}$$

However,

$$\operatorname{sgn}(HB_o)\,D = \operatorname{sgn}(HB_oD) = \operatorname{sgn}(HB) \tag{3-7}$$

Corollary 2 If Y is a realizable admittance matrix, then sgn (A'Y) = 2B-UD.

Proof If Y is realizable then

$$A'Y = HB \tag{3-8}$$

Now suppose that Y is realizable via some A matrix. Then, from the definition of the standard form, there exists a permutation matrix P_2 such that $Y_p = P_2 Y P'_2$ is realizable via A_1 which is in standard form. Let Y_p be partitioned in accordance with A_1 as follows:

$$Y_p = \begin{bmatrix} Y_{K_1} & & & \\ & Y_{K_2} & & \\ & & \ddots & \\ & & & Y_{K_p} \end{bmatrix} = \begin{bmatrix} \hat{Y}_{K_L} \\ \vdots \\ \hat{Y}_{K_p} \end{bmatrix} \tag{3-9}$$

Now consider the product

$$Q = \begin{bmatrix} Q_{K_1} \\ Q_{K_2} \\ \vdots \\ Q_{K_p} \end{bmatrix} = \begin{bmatrix} 1_{K_1} & & & 0 \\ A_1 & 1_{K_2} & & \\ 0 & A_2 & \ddots & \\ 0 & 0 & & \\ \vdots & \vdots & A_{p-1} & 1_{K_p} \end{bmatrix}' \begin{bmatrix} \hat{Y}_{K_1} \\ \vdots \\ \hat{Y}_{K_p} \end{bmatrix} \tag{3-10}$$

Thus,

$$Q_{K_i} = Y_{K_i} + A'_{K_i} \hat{Y}_{K_{i+1}} \quad \text{for } i \neq p \tag{3-11}$$

$$Q_{K_p} = \hat{Y}_{K_p} \tag{3-12}$$

Notice, however, that if Y is realizable

$$Q = HB \tag{3-13}$$

Hence sgn(Q) = 2B-U by Theorem 9. Now $\text{sgn}(Q_{K_p})$ is completely determined by the given Y matrix. Thus, the last K_p rows of B are completely determined by use of Theorem 9. Now utilizing the recursion relation,

$$B_{\ell(\ell-1)} = -A_{\ell-1}, \tag{3-14}$$

A_{p-1} is completely determined. Having determined A_{K-1}, EQ. (3-14) may be used to determine $B_{(K_{p-1})(K_{p-2})}$. The recursion relation may then be utilized to obtain A_{p-2}, and the process may be continued until the entire A matrix is obtained. Having obtained the A matrix, it is a simple matter to determine whether or not $A'Y_pA$ is hyperdominant. Unfortunately this procedure requires a knowledge of the permutation matrix p_2, which is not trivial to obtain. It will be shown that the determination of p_2 is essentially the entire problem. Furthermore, it will be shown that, once p_2 is obtained, a complete synthesis follows in a relatively simple manner.

There is also a simple relation between the signs of the elements of Y and the tree of port voltages. The following theorem was recognized by Guillemin and utilized by Biorci and Civalleri[3, 16, 17].

Theorem 10 If Y is a realizable admittance matrix of nonzero constants, then sgn(Y) is uniquely determined by the tree of port voltages. In fact,

$\text{sgn } y_{ij} = +1$ if ports i and j are oriented in the same direction on the tree path joining them.

$= -1$ otherwise

Proof Consider the unique tree path joining ports i and j. When all ports but i and j are shorted, these two ports have a common node on this tree path. Now, in the two port case with a common node, it is clear that y_{ij} is positive if the ports are oriented oppositely with respect to the common node, and y_{ij} is negative if the two ports are both oriented toward or away from the common node.

Additional properties of the B matrix may be obtained by utilizing this theorem.

One of these properties is a relation between the matrix Y_p and A_1, the A matrix in standard form. Consider the submatrices A_k and the corresponding submatrices Y_k of Y_p. Now the rows of Y_k correspond to ports which form the $(K+1)^{st}$ star, and the columns correspond to the ports which form the K-th star. Thus the (ij)-th element of y_k is only positive if ports i and j are incident at the same node. In this case, the (ij)-th element of A_k is -1. If the (ij)-th element of Y_k is negative then i and j are not incident at the same node. In this case, the (ij)-th element of A_k is zero. Hence

$$A_k = -\left[\frac{\operatorname{sgn}(Y_k) + U}{2}\right] \tag{3-15}$$

Another property of sgn (Y) is contained in the following theorem.

Theroem 11 If all the ports are oriented toward ground and $y_{ij} \neq 0$ for all i and j, then

$$y_{ij} > 0 \quad \text{if i and j are on the same ground path,}$$

and

$$y_{ij} < 0 \text{ otherwise.}$$

Proof If i and j are on the same ground path then they are oriented in the same direction by definition of an orientation toward ground; thus $y_{ij} > 0$ by Theorem 10. If i and j are on distinct ground paths, then the unique path from i to j is part or all of the two respective ground paths. Clearly, i and j are in opposite directions on this path.

This result may also be interpreted in terms of the B matrix.

Theorem 12 Let B_o correspond to an orientation toward ground, and let $Y_1 = [y_{ij}]$ be the corresponding admittance matrix. Let $P = B'B = [p_{ij}]$ be the Gram matrix of B. Then if $y_{ij} \neq 0$,

$$y_{ij} < 0 \text{ for } p_{ij} = 0,$$

and

$$y_{ij} > 0 \text{ for } p_{ij} > 0.$$

Proof Let $\tilde{b}_i$ and $\tilde{b}_j$ denote the i-th and j-th columns of $B_o = [b_{ij}]$ respectively. Suppose

$$\tilde{b}_i'\tilde{b}_j = 0 \tag{3-16}$$

Now all the elements of $\tilde{b}_i$ and $\tilde{b}_j$ are nonnegative. Hence there cannot be an overlap in the nonzero terms of these columns, i.e., if $b_{ki} = 1$ then $b_{kj} = 0$, and if $b_{ki} = 0$ then $b_{kj} = 1$. Thus any ground path which includes port i does not include port j. Therefore, $y_{ij} < 0$ by the previous theorem.

On the other hand if

$$\tilde{b}_i'\tilde{b}_j > 0, \tag{3-17}$$

then there is at least one k such that $b_{ki} = b_{kj} = 1$. Thus at least one ground path includes ports i and j. Therefore, $y_{ij} > 0$ by the previous theorem.

Besides orienting all ports toward ground, it is convenient to assign a standard numbering of nodes. A node numbering of a tree will be called an adjacent numbering if when all ports are directed toward ground the i-th node is taken as the node from which port is is incident away. It is easily seen that this corresponds to a unique numbering. Indeed suppose ports i and j were incident away from node k. Then there would exist two ground paths from node k. Since it was assumed that all ports were directed toward ground, this is certainly impossible. This labelling of nodes and orientation of ports implies a simple relation between the B matrix and sgn (Y).

Theorem 13 Let $y_{ij} \neq 0$ for all i and j. Assume that Y is realizable, and B corresponds to a tree of port voltages which has been oriented toward ground and numbered adjacently. Then

$$B + B' = \frac{\text{sgn } Y + U}{2} + 1_N$$

where U is an n x n matrix, all of whose elements are +1, and 1_N is the N-th order unit matrix.

Proof Suppose $i \neq j$ and $y_{ij} > 0$. Thus ports i and j are on the same ground path. Since the nodes are numbered adjacently, either the ground path from node j includes port i, or the ground path from node i includes port j but not both. Thus $b_{ij} = 1$ and $b_{ji} = 0$ or $b_{ji} = 1$ and $b_{ij} = 0$. Now let $y_{ij} < 0$, for $i \neq j$. Since the nodes are numbered adjacently, ports i and j must be on distinct ground paths. Hence $b_{ij} = 0$ and $b_{ji} = 0$. Finally let $i = j$; clearly $y_{ii} > 0$. Since the nodes are numbered adjacently, the ground path from node i includes port i, and hence, $b_{ii} = 1$. This completes the proof.

A synthesis procedure involves solving the matrix equation of theorem 6 for B when sgn (Y) is given. Now if Y is known, a priori, to correspond to a lower triangular B, the solution for B is immediate. In fact, when B is lower triangular, it may be obtained from B + B' by taking 1/2 of the diagonal terms and reducing all elements above the diagonal to zero. The proof is left to the reader.

It will be shown in the next section that B may be rendered lower triangular if the ports corresponding to the minimum mutual admittance are obtained.

3.3 The Minimum Mutual Admittance

Definition 15 Two ports are said to be in series if they are both incident at a common node of degree two.

Theorem 14 [3, 18] If ports i and j are in series and there are no zero elements in Y, then

$$y_{ij}y_{ki}y_{kj} > 0 \text{ for all } k \neq i, j$$

Proof Since i and j are in series, they define a partition of the tree into two distinct subtrees T_1 and T_2; see Fig. 5.

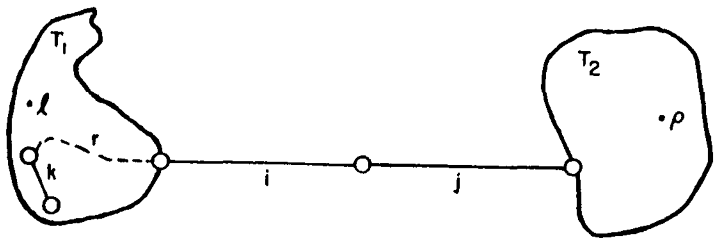


Fig. 5 Series ports

T_1 and T_2 are distinct since any path between nodes ℓ and ρ which did not include i and j would yield a closed path in the original tree. Consider any port k in T. There must be a unique tree path r from port i to port k which does not include port j. Now the unique tree path from port j to port k must therefore be identical with r except for the addition of port j. Hence y_{ki} and y_{kj} have the same sign and y_{ij} is positive if i and j are oriented in the same direction. Similarly, y_{ki} and y_{kj} have opposite signs and y_{ij} is negative if i and j are oriented in opposite directions. Hence,

$$y_{ij}\, y_{ki}\, y_{kj} > 0 \tag{3-18}$$

The same argument applies if port k is in B.

Suppose the tree of port voltages consists entirely of series ports. In this case, the tree is said to be a linear tree. As an example of the linear tree, consider a fourth order Y which corresponds to a linear tree; see Fig. 6

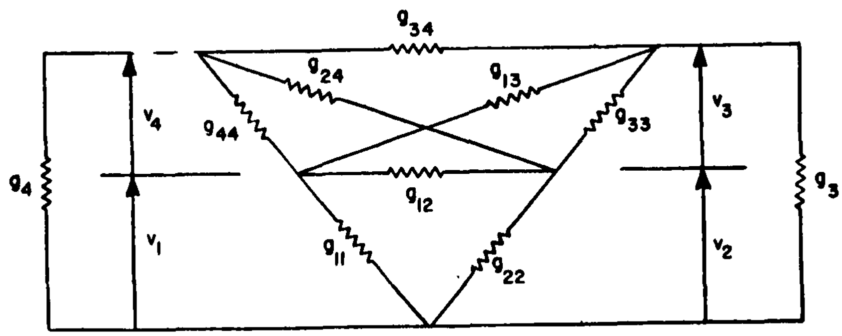


Fig. 6 A fourth order linear tree

$$
\begin{aligned}
-Y_{12} &= g_{12} + g_{13} + g_{24} + g_{34} \\
-Y_{13} &= g_{13} + g_{34} \\
Y_{14} &= g_{34} + g_4 + g_{24} \\
Y_{23} &= g_3 + g_{13} + g_{34} \\
-Y_{24} &= g_{24} + g_{34} \\
-Y_{34} &= g_{34}
\end{aligned}
\tag{3-19}
$$

Notice that this network has the property that if the graph is complete, then $|y_{34}|$ is strictly less than the magnitude of any other mutual admittance. This property may be easily extended to the general case by observing that ports 3 and 4 have the property that they are both incident at a node which is of exactly degree 1 in the tree of port voltages.

Definition 16 A node which is of degree one in a tree is said to be a tip node. The port which is incident at a tip node is called a tip port. Obviously, any tree has at least two tip ports by lemma 1 of section 2.0.

Theorem 15 If i and j are not both tip ports, then in any complete graph $|y_{ij}|$ can not be the minimum magnitude of all the mutual admittances. Therefore, the minimum of the magnitude of the mutual admittances must correspond to two tip ports.

Proof Suppose i and j are not tip ports, and consider the determination of $|y_{ij}|$. First, when all ports but i and j are shorted, i and j have a common node. Let the common node, which is found by obtaining the unique tree path from i to j, be represented as in Fig. 7.

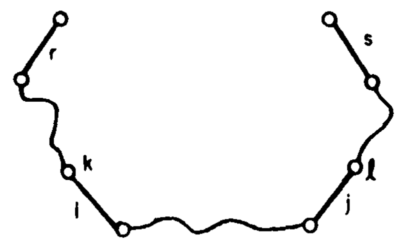


Fig. 7 Relation between tip ports and non-tip ports.

If i is not a tip port then node k can not be of degree one and must, therefore, have a tree path to some tip port r. Likewise, there is a tree path from port j to tip port s. Clearly ports i, j, r, and s form a fourth order linear tree. Then by the previous example,

$$|y_{rs}| < |y_{ij}| \tag{3-20}$$

if the graph is complete and i and j are not both tip ports. Hence $|y_{ij}|$ can not be a minimum. The proof of theorem 15 may be extended to thecase of a graph which is not complete. Again it is true that there is no mutual admittance corresponding to two non-tip-ports which is less than the mutual admittance between some pair of tip-ports. However, they may be equal. Thus if the graph is not complete and the minimum magnitude element is not unique, then at least one of these elements corresponds to a pair of tip ports but not necessarily all of these elements. However, the ports corresponding to minimum magnitude elements which are not tip ports must both be on a tree path between two tip ports that do correspond to a minimum. Thus all the minimum magnitude elements correspond to tip ports if and only if the subtree corresponding to these minimum elements is a star. It will be shown in a subsequent section that the tree of port voltages of a matrix Y with no zero elements is a star if and only if there is a D such that DYD is hyperdominant; if Y has this property it will be called a matrix of hyperdominant form.

An example of the application of theorem 15, consider a Y matrix whose off-diagonal terms are all positive. It is well known that the only possible tree in this case is the linear tree. Let y_{ij} be the minimum off-diagonal term. Thus ports i and j are tip ports. Now if port i is shorted, i.e., row and column i are deleted from Y, port j is still a tip port and the only other tip port in this subtree is the port k which was adjacent to i. Therefore, when i is shorted the minimum element must occur in the j^{th} column again. This process may be continued until the order of the ports is established. In fact, the order of the ports is exactly that of the order of the sizes of the elements in the j^{th} column. Notice that this completely avoids the usual permutation of Y into the linearly tapered form[4]. Since the order of the ports is known, the A matrix is obtained by inspection, and the congruence transformation A'YA may be performed to determine if it leads to a hyperdominant matrix.

If the linear tree is not complete, viz, the mimimum magnitude element is not unique, then the procedure may be extended as follows: the tip port may be taken as the port which corresponds to the maximum number of mimimum elements. Indeed suppose i is not a tip port and it corresponds to a port with a maximum number of minimum elements. Now if there is a non tip port j such that $|y_{ij}|$ is a minimum, then there are two tip ports k and ℓ such that $|y_{k\ell}|$ is also a minimum. Certainly ports i, j, k, and ℓ form a fourth order linear tree when all other ports are shorted; it follows from the discussion of the linear tree that i may also be taken as a tip port. If there is no non tip port j such that $|y_{ij}|$ is a minimum, then there is a tip port k such that $|y_{ik}|$ is a minimum. Let ℓ be the other tip port in the tree, and consider the third order linear tree obtained by shorting all ports but i, k, and ℓ. Certainly $|y_{k\ell}|$ is a minimum. Thus if $|y_{i\ell}|$ is not a minimum, then port k is minimum with respect to more ports than j which contradicts the assumption. Alternatively if $|y_{i\ell}|$ is a mini-

mum, then the order of i, k, and ℓ is immaterial, and i may also be taken as a tip port.

The properties of the linear tree may be used to determine the order of a set of series ports in an arbitrary tree. It is convenient at this point to introduce the concept of distance in a tree.

Definition 17 Ports r and k are said to be distance ℓ from each other in the tree of port voltages if the unique tree path from k to r contains ℓ ports besides r and k. The zero distance corresponds to r and k incident at a common node.

Suppose ports $l_1, l_2, \ldots, l_r$ are in series. Let port t be an arbitrary tip port in the tree. Then the tree formed by shorting all but t, $l_1, l_2, \ldots, l_r$ must be linear since all the nodes in the set of series ports are of degree two except for the tip nodes of the series set of ports. Thus by inspection of the Y matrix the series ports may be ordered in increasing distance from the tip port.

Another method for determining tip ports was given by Biorci, Cederbaum, et. al.[3, 18]

Definition 18 A set of tip ports are said to be star-connected tip ports if they are all incident at a common node.

Theorem 16 If i and j are star-connected tip ports corresponding to a complete graph, then $y_{ij}\, y_{ki}\, y_{kj} < 0$ for all $k \neq i, j$.

Proof Since i and j are star connected they are incident at a node n; see Fig. 8

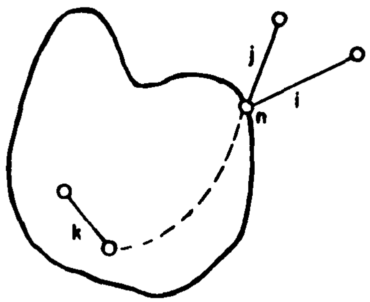


Fig. 8 Tip-port sign test

Let k be any port other than i or j. There is a unique tree path from node n to k. Thus y_{ik} and y_{jk} have the same sign if ports i and j are oriented negatively with respect to each other. Similarly, y_{ik} and y_{jk} have opposite signs if they are oriented positively with respect to each other. This completes the proof.

It has been shown that there always exists a pair of star-connected tip ports in a tree which contains no series ports[18]. A much simpler proof is given in theorem 17.

Theorem 17 There always exists a pair of star connected tip ports in a tree with no series ports.

Proof Consider the longest path in the tree. Clearly the initial and terminal nodes in the longest path are tip nodes. Now the node adjacent to either one of these tip nodes can not be of degree two since there are no series ports. Hence there is at least one port, incident at the node adjacent to a tip node, which is not contained in the longest path. If this port is not a tip port, then the original path is not the longest path. Therefore, there is a least one pair of star connected tip ports. This argument may also be applied to the other tip node in the longest path if it is assumed that the tree has at least four ports, and the conclusion is that there is another pair of star connected tip ports.

3.4 Synthesis Procedure for a Complete Graph

The synthesis procedure will be restricted to a complete graph since this is the simplest case; applications to non-complete graphs are discussed in the next section.

First suppose a tip port k has been chosen by applying theorem 15. The tip node will be chosen as ground and all ports will be directed toward ground. This is accomplished by DYD where D has the property of rendering all y_{ki} positive. Now let N_i be the number of ports including port i which are oriented in the same direction as port i; N_i is the number of positive elements in the i^{th} column of DYD, or the number of ports included in some ground path through port i. It is assumed that any tree has been endowed (mentally) with an adjacent node numbering. Furthermore, if ports $i_1, i_2, \ldots, i_r$ are in series then let them be ordered such that $i_\ell < i_\rho$ if i_ℓ is closer to k than i_ρ. In this case, it is easy to determine a permutation of Y which must correspond to a lower triangular B matrix.

Theorem 18 Let all ports be oriented toward a tip node and assume that the nodes are numbered adjacently. If Y is permuted (ports renumbered) such that $i<j$ if $N_i \geq N_j$ and series ports ordered in increasing distance from the tip node, then B must be lower-triangular.

Proof Suppose B is not lower triangular. Then there is a first row in B which destroys the lower triangular form. Let ρ be the column in which the nonzero entry of the n^{th} row occurs. Now by assumption $\rho > n$. However, the ground path from node n includes port ρ. By the adjacent node numbering, the ground path from port n includes port ρ. Thus ρ is closer to ground than n. This is impossible if ρ and n are in series; hence, ρ and n are not in series. If ρ and n are not in series then there is at least one ground path which includes port ρ but does not include port n. On the other hand, every ground path which includes n also includes ρ. Therefore N_ρ is greater than N_n, but this is in contradiction to the assumed order for the ports. Hence, B must be lower triangular.

A simple synthesis procedure for an n^{th} order Y matrix on a complete graph of n + 1 nodes may be based on theorems 18 and 12. The steps are outlines below.

1. Determine a tip-port by finding the minimum magnitude element.
2. Permute the tip-port to the first row and column and re-orient all ports so that y_{kl} is positive for all k.
3. Determine N_i for all i by counting the number of positive elements in the i^{th} column.
4. If N_i and N_j are equal for some unequal i and j, then determine whether or not ports i and j are in series by the sign test.
5. Order the columns of Y so that $N_i \leq N_j$ for all $j < i$.
6. If there are any series ports, then order them in accordance as their distance from the ground node increases. This may be accomplished by applying the linear tree realization procedure to the submatrices consisting of the ground port and each set of series ports.
7. B may now be assumed to be lower triangular, and the equation

$$B + B' = \frac{\text{sgn}\,(Y_o) + U}{2} + 1_N \tag{3-21}$$

   where Y_o is the matrix resulting from the first 5 steps, may be easily solved for B.
8. Determine the tree corresponding to B, if there is one, by utilizing the topological interpretation of B. If this B does not correspond to a tree, then Y is not realizable. If there is a tree, A may be obtained from the tree by inspection. Note that finding the tree from B is usually easier than obtaining the inverse of B algebraically.
9. Perform the Kron transformation $A'Y_oA$, and synthesize the result as a nodal admittance matrix if it is hyperdominant. If it is not hyperdominant, then Y is not realizable.

3.5 Synthesis Procedure for an Incomplete Graph

From the discussion of the linear tree, it is evident that series ports do not present a problem in the incomplete case. If sgn (Y) is well defined, then the only problem in the previous synthesis procedure is obtaining a tip port, however, the star-connected tip port test may be utilized. Alternatively, the minimum magnitude procedure may be extended in a simple fashion. Suppose the minimum magnitude element is not unique and that the graph obtained by shorting all ports but those which define minimum magnitude elements is not a star, i.e., the corresponding submatrix is not of hyperdominant form. It will be shown that this tree, although not a star, retains a relatively simple structure.

Definition 19 A series star is a tree in which there is only one node of degree larger than two.

Theorem 19 Suppose $y_{ij} \neq 0$ for all i and j. The tree obtained by shorting all ports but the minimum ports is a series star.

Proof Consider the tree of minimum ports. Suppose there are at least two nodes of degree greater than two. Let i and j be the two nodes of degree greater than two which are connected by the shortest path R; see Fig. 9.

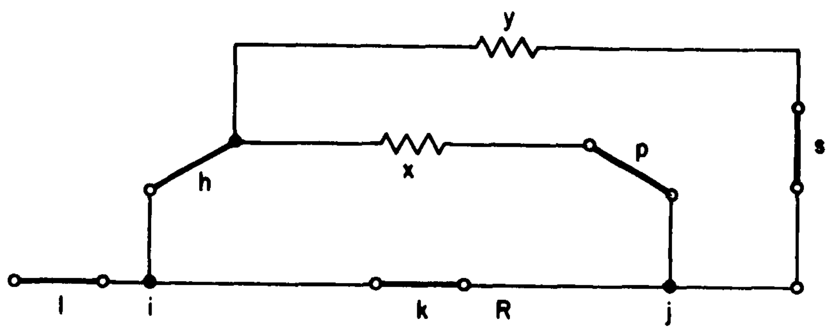


Fig. 9 The tree of minimum port

Thus path R consists entirely of series ports; let one of these series ports be k. By reference to the discussion following theorem 15, it can be seen that k lies on a path between two tip ports, say h and p, which are also minimum ports. Certainly h and p, are not contained in path R. Let x denote the minimum magnitude; hence

$$|y_{ph}| = x, \tag{3-22}$$

and either $|y_{kh}|$ or $|y_{kp}|$ is equal to x. Therefore, the conductance between ports p and h is equal to x. Furthermore since i and j are not of degree two, there are paths from i and j to two tip ports, say ℓ and s, which are also minimum ports, and distinct from h and p. Let the conductance between h and s be y. Then

$$|y_{hs}| = g \tag{3-23}$$

and

$$|y_{kh}| \geq x + y \tag{3-24}$$

Since y can not be zero - there are no zeros in the original matrix - $|y_{kh}|$ can not be a minimum. Likewise if $|y_{\ell p}|$ is not zero, then $|y_{kp}|$ can not be a minimum. This is a contradiction; hence either node i or j must be of degree two or less.

Thus if the tree of minimum ports is not a star, then it must be a series star. Certainly, a tip port in a series star is one of the series ports. This tip port may be

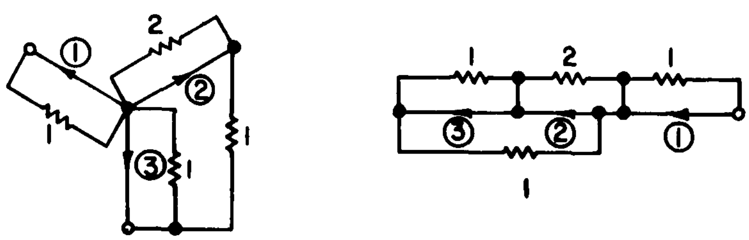
obtained by utilizing the test for a tip-port of a linear tree. The procedure is outlined below.

1. Consider any set of series ports in the submatrix corresponding to the minimum elements if this submatrix does not correspond to a star. The existence of a set of series ports is guaranteed by theorem 18.
2. Determine the tip ports of these series ports which form a linear tree if the minimum magnitude element of the linear tree is unique. Clearly these are tip ports of the tree of minimum ports, and therefore they are also tip ports of the original tree.
3. If the linear subtree does not have a unique pair of tip ports, then the procedure for obtaining a tip port of an incomplete linear tree may be utilized to obtain a tip port of the original tree.
4. Utilizing the tip port, the synthesis proceeds via theorem 17; the techniques of the linear tree realization are used to determine the order of series ports.

Thus in the incomplete case, the tree of port voltages may also be obtained by a simple procedure. Notice that in the incomplete case thetree may not be unique. However, the only non-uniqueness is in the ordering of series ports.

3.6 Uniqueness

It has been pointed out by Cederbaum that a non-unique realization may exist when there are elements of Y which equal zero[18]. This may be easily observed in the example shown in Fig. 10.

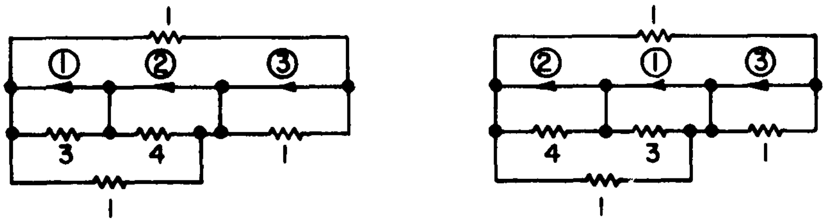


$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Fig. 10 A Y matrix with two distinct realizations

The result is obvious in this case since the first port is completely isolated from all the others by virtue of the zero mutual admittance ; it may therefore be included at any node. Thus when there are zeros present there may be more than one tree of port voltages where these trees are not merely permutations of each other.

Non-uniqueness is also possible in the case with no zeros elements of Y if the realization is not on a complete graph. This is demonstrated by the example shown in Fig. 11. Notice that in this case the non-uniqueness is manifested in the ordering of the ports. Furthermore this reordering of the ports does not correspond to a mere interchange of two identical rows of Y. From the previous section, it is evident that the only non-uniqueness in the case of no zero elements is in the ordering of series ports.



$$y = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Fig. 11 A non-unique realization when there are no zero elements

On the other hand, the realization is always unique if the graph is complete. First observe that the tip port is uniquely determined in this case. Now the equation

$$B + B' = \frac{\mathrm{sgn}\,(Y) + U}{2} + 1_n \tag{3-25}$$

implies that B is uniquely determined from sgn (Y) if B is known a' priori to be lower triangular. If a tip port is known, then theorem 17 specifies a unique procedure for obtaining the permutation of Y that corresponds to a lower triangular B when N_i and N_j are unequal for all i not equal to j. Suppose that they are equal. If i and j are in series, then there is only one order of i and j which corresponds to a lower triangular B, and this order is determined uniquely in the complete graph case by application of the linear tree techniques. If i and j are not in series then they must lie on distinct ground paths since N_i is equal to N_j. In this case, theorem 17 implies that B is lower triangular for either order of i and j. Assume that

$$i \doteq j + 1 \tag{3-26}$$

Since i and j are on distinct ground paths,

$$b_{ji} = 0 \tag{3-27}$$

Thus interchanging the i^{th} and j^{th} rows and columns of B does not alter the lower triangular property, and it corresponds to interchanging the i^{th} and j^{th} rows and columns of Y. Therefore, the tree of port voltages is uniquely determined by sgn (Y). Since the tree is uniquely determined , the realization is unique by the Kron transformation.

As an example of uniqueness in the noncomplete case, consider a hyperdominant Y with no zero elements. (Y may be zero in the sum of the columns). One of the rows must correspond to a tip port. Thus, without loss of generality, assume it is the first row. All ports are directed toward the tip node by

$$D = \mathrm{diag}\ (+1,\ -1,\ -1,\ \ldots,\ -1) \tag{3-28}$$

which yields exactly two positive elements in each column. It is easily verified that B is lower triangular with +1 elements along the diagonal, +1 elements in every row of the first column, and zero elsewhere. Clearly, this B corresponds to a star of port voltages. Hence, the realization of a hyperdominant matrix without zero elements is unique.

4.0 Examples of the Synthesis Procedure

A few examples are given in this section to illustrate the simplicity of the synthesis. The synthesis technique is also applied to the realization of cut-set matrices. The procedure is simplified in the case of the cut-set matrix since the Kron transformation is not necessary; using the connection matrix, the condition for realizability is that CA be an incidence matrix. The incidence matrix may be tested for realizability by inspection and the graph is synthesized immediately.

4.1 Realization of a Tenth Order Admittance Matrix

Consider the Y matrix given by Civalleri[14].

$$
Y = \begin{bmatrix}
67 & 20 & 10 & 15 & 45 & 17 & 23 & 33 & 15 & 26 \\
20 & 30 & 3 & -6 & 23 & 5 & 12 & 8 & 3 & 29 \\
10 & 3 & 30 & 2 & 5 & -4 & -1 & 11 & 5 & 3 \\
15 & -6 & 2 & 45 & 30 & 9 & 4 & 11 & -15 & -12 \\
45 & 23 & 5 & 30 & 79 & 14 & 16 & 25 & -18 & 29 \\
17 & 5 & -4 & 9 & 14 & 30 & -7 & 24 & 3 & 5 \\
23 & 12 & -1 & 4 & 16 & -7 & 49 & -8 & 7 & 12 \\
33 & 8 & 11 & 11 & 25 & 24 & -8 & 42 & 8 & 14 \\
15 & -3 & 5 & -15 & -18 & 3 & 7 & 8 & 36 & -3 \\
26 & 29 & 3 & -12 & 29 & 5 & 12 & 14 & -3 & 41
\end{bmatrix} \qquad (4\text{-}1)
$$

Certainly, ports 3 and 7 are tip ports, and ports 2 and 10 are in series. Now the submatrix corresponding to ports 3, 2, and 10 corresponds to a noncomplete graph since it pertains to a linear tree and it conatins two equal elements. However, this does not invalidate the procedure since a tip port has been obtained. The simplest approach is, probably, to short out one of the series ports and continue the synthesis on the resulting 9-port. Let port 10 be shorted, i.e., delete row and column 10. Now let the tip node of port 3 be chosen as ground and orient all ports toward ground by changing the directions of ports 6 and 7. Now arrange the columns in decreasing number of positive elements; the appropriate permutation is

$$123456789 \longrightarrow 361549827$$

This yields

$$
Y = \begin{bmatrix}
30 & 11 & 10 & 5 & 3 & 2 & 5 & 1 & 4 \\
11 & 42 & 33 & 25 & 8 & 11 & 8 & 8 & -24 \\
10 & 33 & 67 & 45 & 20 & 15 & 15 & -23 & -17 \\
5 & 25 & 45 & 79 & 23 & 30 & -18 & -16 & -14 \\
3 & 8 & 20 & 23 & 30 & -6 & -3 & -12 & -5 \\
2 & 11 & 15 & 30 & -6 & 45 & -15 & -4 & -9 \\
5 & 8 & 15 & -18 & -3 & -15 & 36 & -7 & -3 \\
1 & 8 & -23 & -16 & -12 & -4 & -7 & 49 & -7 \\
4 & -24 & -17 & -14 & -5 & -9 & -3 & -7 & 30
\end{bmatrix} \qquad (4\text{-}2)
$$

The corresponding B is obtained by placing all +1 elements along the diagonal and below the diagonal a +1 when y_{ij} is positive and a zero when y_{ij} is negative. Thus

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (4\text{-}3)$$

The tree is then

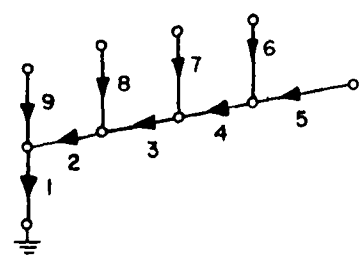


Fig. 12 Tree for standard form of example 1

The tree of the original Y is obtained by reversing the sign changes and the permutation.

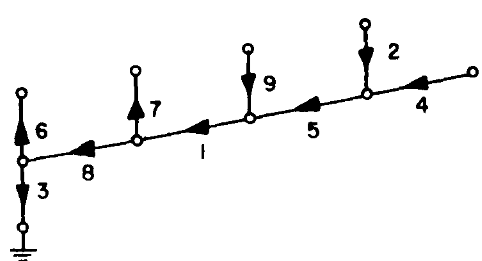


Fig. 13 Tree for example 1 with port 10 excluded

Port 10 may now be included, and its position may be found by considering the linear tree of ports 2, 10, and 8. Of these ports, ports 2 and 9 yield the minimum elements; hence port 2 is a tip port when port 10 is included. Thus the complete tree is as follows.

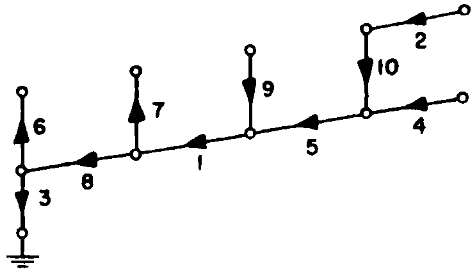


Fig. 14 Tree for example 1 with port 10 included

The matrix A may now be obtained by inspection and the transformation A'YA is easily performed; it is not given here since this aspect is not pertinent to the techniques described in the report.

4.2 Determination of the Tree from a Fifteenth Order sgn(Y)

Consider the sgn Y matrix given by Biorci and Civalleri[2].

$$\mathrm{sgn}\,(Y) = \begin{bmatrix}
+1 & -1 & -1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 \\
-1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\
-1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 \\
-1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & -1 \\
-1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & -1 & -1 & -1 \\
+1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 & +1 & +1 & -1 & +1 & +1 & -1 & -1 \\
-1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & +1 & -1 & -1 & -1 & +1 \\
+1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 \\
+1 & -1 & -1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 \\
-1 & -1 & +1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 & +1 & +1 \\
-1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 & +1 \\
-1 & +1 & +1 & +1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 & -1 & +1 & +1 \\
+1 & -1 & +1 & +1 & -1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & +1 & +1 & +1 \\
+1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 \\
-1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 & +1
\end{bmatrix} \tag{4-4}$$

Since the magnitudes are not given, the test for star-connected tip ports must be used. Ports 7 and 8 are found to be star-connected tip ports. Port 7 is chosen as the grounded port, and all ports are oriented toward ground by reversing the directions of ports 1, 9, 10, 12, 12, and 14.

$$\begin{bmatrix}
+1 & +1 & +1 & +1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & +1 & +1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 \\
+1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 \\
+1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 \\
+1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 \\
-1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
-1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & -1 & +1 & -1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 \\
-1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 \\
+1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 \\
-1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 & +1 & -1 \\
+1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 \\
-1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 & +1 & -1 \\
+1 & +1 & -1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & +1
\end{bmatrix} \tag{4-5}$$

The result of counting the number of +1 elements in each column is given in the following table.

i	N_i
1	11
2	13
3	7
4	7
5	8
6	4
7	15
8	2
9	12
10	5
11	8
12	3
13	7
14	14
15	7

Therefore, the following permutation will render B lower triangular

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15 \longrightarrow 7\ 14\ 2\ 9\ 1\ 5\ 11\ 15\ 3\ 4\ 13\ 10\ 6\ 12\ 8 \tag{4-6}$$

It can be shown that there are no series ports by the sign test; however, since the magnitudes are not given, the order of the series ports is immaterial. Performing the indicated permutation yields

$$\begin{bmatrix}
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & -1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 \\
+1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 \\
+1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 \\
+1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1
\end{bmatrix} \tag{4-7}$$

It is not necessary to give the B matrix explicity since the relation of B to this form of Y is quite simple. Indeed, it is clear from examination of the above matrix that the tree is as shown below.

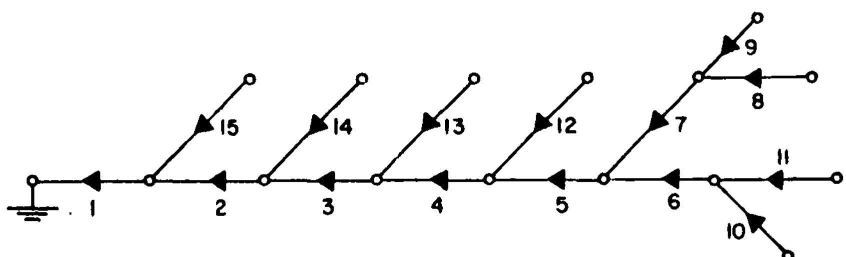


Fig. 15 Tree for standard form of example 2

Reversing the permutation and orienting the ports as they were originally yields the following tree for the given sgn (Y).

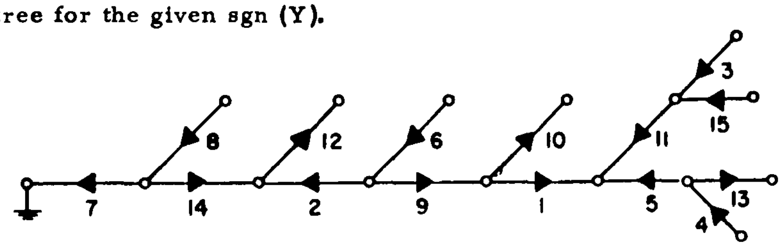


Fig. 16 Tree for example 2

4.3 Realization of a Cut-Set Matrix

The fact that the realization of a fundamental cut-set or fundamental circuit matrix is equivalent to therealization of a short-circuit n x n admittance matrix on n +1 terminals has been recognized by many authors[17, 19].

Consider the cut-set matrix given by Halkias and Kim.[19].

$$C'_f = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \end{bmatrix} \tag{4-8}$$

In this case n + 5 and the number of arcs in the graph corresponding to C_f is 15. Since $(n + 1)n/2$ is 15 in this case, the graph is complete. The realization of C_f is accomplished by realizing

$$Y = C'_f\, C_f \tag{4-9}$$

as a short-circuit admittance on an (n + 1) terminal graph of unit conductances. Now

$$C'_f\, C_f = Y = \begin{bmatrix} 9 & 3 & 3 & 3 & 3 \\ 3 & 5 & 1 & -1 & 1 \\ 3 & 1 & 5 & 1 & -1 \\ 3 & -1 & 1 & 5 & 1 \\ 3 & 1 & -1 & 1 & 5 \end{bmatrix} \tag{4-10}$$

There are six minimum magnitude elements in this Y. If all ports but those corresponding to the minimum elements are shorted, then the resulting submatrix is of the form DHD, where H is hyperdominant and

$$D = \text{diag. } (1, -1, 1-1) \tag{4-11}$$

Therefore, ports 2, 3, 4, and 5 correspond to a star when port 1 is shorted. From the proof of theorem 7 it is clear that all of the ports 2, 3, 4, and 5 are tip-ports. Alternatively, it was known from C_f that the graph was complete, and this conclusion is then immediate from theorem 7. Choosing port 5 as a tip port, directing all ports toward the tip node, and arranging Y is order of decreasing number of positive elements in the columns yields the following matrix.

$$\begin{bmatrix} 5 & 3 & 1 & 1 & 1 \\ 3 & 9 & -3 & 3 & 3 \\ 1 & -3 & 5 & -1 & -1 \\ 1 & 3 & -1 & 5 & -1 \\ 1 & 3 & -1 & -1 & 5 \end{bmatrix} \tag{4-12}$$

The tree corresponding to this matrix is shown below

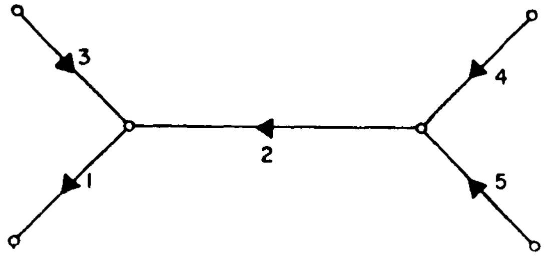


Fig. 17 Tree for example 3

It is convenient to redefine ground at a node of larger degree. In this case,

$$\mathbf{A'YA} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 5 & 3 & 1 & 1 & 1 \\ 3 & 9 & -3 & 3 & 3 \\ 1 & -3 & 5 & -1 & -1 \\ 1 & 3 & -1 & 5 & -1 \\ 1 & 3 & -1 & -1 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & -1 & 5 \end{bmatrix} \tag{4-13}$$

Clearly this short-circuit admittance matrix corresponds to a network of positive unit conductances.

4.4 Realization of Another Cut-Set Matrix

Consider the fundamental cut-set matrix

$$C_f' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix} \tag{4-14}$$

$$Y = C_f' C_f = \begin{bmatrix} -4 & -1 & -2 & -1 \\ -1 & 4 & -2 & -1 \\ -2 & -2 & 6 & 3 \\ -1 & -1 & 3 & 4 \end{bmatrix} \tag{4-15}$$

Ports 1, 2, and 4 correspond to minimum elements. If port 3 is shorted, then the corresponding submatrix is hyperdominant. Hence all these ports are tip ports. Choose port 1 as a tip port, and reverse the direction of port 1. The necessary permutation is then 1 2 3 4 $\longrightarrow$ 1 4 3 2. This yields

$$\begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 4 & 3 & -1 \\ 2 & 3 & 6 & -2 \\ 1 & -1 & -2 & 4 \end{bmatrix} \tag{4-16}$$

The corresponding B matrix is

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{4-17}$$

Now ports 2 and 3 are in series. Port 3 is certainly closer to ground. Hence the tree appropriate to the original matrix is

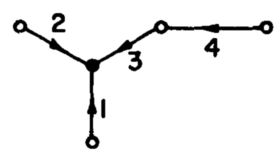


Fig. 18 Tree for cut-set matrix of example 4

The product $C_f A$ must be the incidence matrix of the graph.

$$(C_f A)' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix} = S' \tag{4-18}$$

The graph is shown below

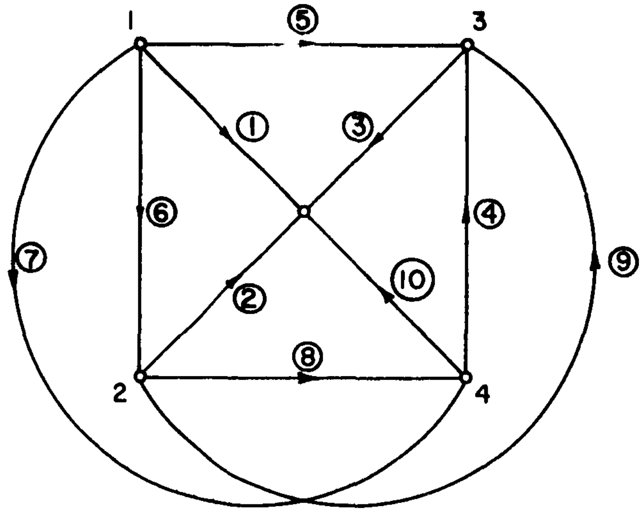


Fig. 19 Graph for cut-set matrix of example 4

4.5 Realization of a Cut-Set Matrix on an Incomplete Graph

Consider the following fundamental cut-set matrix.

$$C_f' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad (4\text{-}19)$$

Then

$$Y = C_f'C_f = \begin{bmatrix} 3 & -2 & -1 & -1 & 1 \\ -2 & 4 & 1 & 2 & -1 \\ -1 & 1 & 3 & -1 & 1 \\ -1 & 2 & -1 & 4 & -2 \\ 1 & -1 & 1 & -2 & 3 \end{bmatrix} \qquad (4\text{-}20)$$

Now all ports correspond to some minimum element. However, the matrix is not of hyperdominant form. Now the tree obtained by shorting port three is linear, i.e., ports 1, 2, 3 and 5 are in series when 3 is shorted. On the other hand, port 3 is certainly not in series with either 1, 2, 4, or 5. Certainly port 1 is a tip port of the linear tree obtained by shorting port 3. Thus port 1 can be taken as a tip - port. Furthermore port 4 is closer to ground than port 5. The ports are directed toward

ground by reversing the directions of ports 1 and 5. The permutation

$$12345 \longrightarrow 12543 \tag{4-21}$$

yields the proper order for the columns, and the resulting standard form is

$$\begin{bmatrix} 3 & 2 & 1 & 1 & 1 \\ 2 & 4 & 1 & 2 & 1 \\ 1 & 1 & 3 & 2 & -1 \\ 1 & 2 & 2 & 4 & -1 \\ 1 & 1 & -1 & -1 & 3 \end{bmatrix} \tag{4-22}$$

The corresponding B matrix is

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{4-23}$$

The tree appropriate to the original Y matrix is shown in Fig. 20

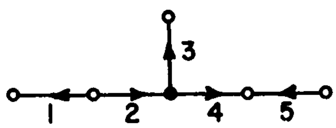


Fig. 20 Tree for example 5

Hence the A matrix is

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{4-24}$$

Then

$$(C_f A)' = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} = S' \tag{4-25}$$

The matrix S' is certainly a reduced incidence matrix; its graph is shown in Fig. 21.

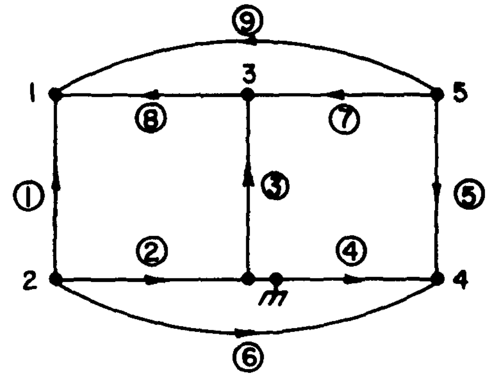


Fig. 21 Graph for cut-set matrix of example 5

4.6 Synthesis of a 3 Port on 4 Nodes

It is relatively simple to establish the conditions for the realization of an arbitrary 3 port on 4 terminals. In this case there are only two trees which are not permutations of each other, i.e., two non-isomorphic trees. The corresponding connection matrices are

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{4-26}$$

In order that a given Y be realizable with A_1 it is obvious that it must be dominant and capable of transformations to hyperdominant by -1 multiplications. For reasons that will become apparent, it is convenient to assume that Y is in a standard form defined as follows:

$$y_{23} \leq 0 \quad , \quad y_{13} \geq 0 \tag{4-27}$$

and

$$|\, y_{13}| \geq |\, y_{23}| \tag{4-28}$$

If Y is not in this form it may always be placed in this form by row and column operations, and multiplications by -1. Also when Y is in the standard form, the sign of y_{12} may not be changed without destroying the standard form. Now suppose that $y_{12} > 0$. Then Y is of hyperdominant form and can only be realized with a tree of port voltages corresponding to A_1. Therefore, if $y_{12} > 0$, the necessary and sufficient condition for realization is that Y be dominant.

Alternatively, suppose $y_{12} \leq 0$; certainly Y cannot be rendered hyperdominant by only -1 multiplications. If it is realizable it must be accomplished via A_2. Performing the indicated operations it is seen that,

$$A_2'YA_2 = \hat{Y} = \begin{bmatrix} y_{11}+y_{33}-2y_{13} & y_{12}-y_{23} & -(y_{33}-y_{13}) \\ y_{12}-y_{23} & y_{22} & y_{23} \\ -(y_{33}-y_{13}) & y_{23} & y_{33} \end{bmatrix} \tag{4-29}$$

Now since paramountcy is certainly necessary, it will be assumed that

$$y_{33} \geq | y_{13} | \tag{4-30}$$

Therefore, all the elements in the 3^{rd} row of $\hat{Y}$ are non-positive (it is again assumed that Y is in standard form). The condition for dominance of the last row is

$$y_{33} \geq | y_{23} | + | y_{33}-y_{13}| \tag{4-31}$$

or

$$0 \geq | y_{23} | - | y_{13}| \quad , \tag{4-32}$$

which is always insured by the standard form. The last row is, therefore, always hyperdominant. Notice that the signs of $\hat{Y}$ may not be altered in any manner without destroying its possible hyperdominant nature. It is therefore necessary that

$$|y_{23} | \leq | y_{12} | \tag{4-33}$$

Dominance of the second row is then obtained by

$$y_{22} \geq | y_{12}| \tag{4-34}$$

Hyperdominance is then insured if the first row is dominant, i.e.,

$$y_{11} + | y_{23} | \geq | y_{13}| + | y_{12} | \tag{4-35}$$

The possibility remains that a permutation of Y be realizable in the case of $y_{12} \leq 0$. Investigation of the remaining five permutations of Y yields that the only other possibility of realization is that the following inequalities be satisfied:

$$| y_{23}| \leq | y_{12}| \tag{4-36}$$

and

$$y_{11} + | y_{23}| \geq |y_{13} | + y_{12}|, \tag{4-37}$$

the realization of the original Y being accomplished via

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4-38}$$

These results are summarized in the following theorem.

Theorem 20 The necessary and sufficient conditions that a constant symmetric three by three matrix Y with diagonal elements positive and not less than any element in the same row be realizable as the short-circuit admittance matrix of a 3-port defined on four terminals are that in standard form, viz., $y_{23} < 0$, $y_{13} \geq 0$, and $|y_{13}| \geq |y_{23}|$

$$y_{12} \geq 0 \text{ and Y dominant} \tag{4-39}$$

or

$$y_{12} \leq 0, \tag{4-40}$$

and

$$y_{11} + |y_{23}| \geq |y_{13}| + |y_{12}|, \tag{4-41}$$

and

$$y_{33} + |y_{12}| \geq |y_{23}| + |y_{13}| \tag{4-42}$$

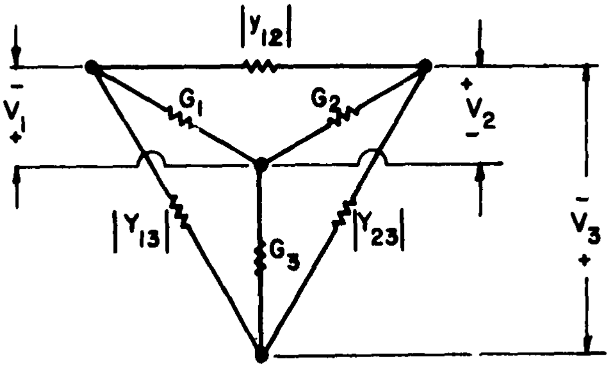
If $|y_{23}| \leq |y_{12}|$, the realization is accomplished by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & +1 \end{bmatrix} \tag{4-43}$$

If $|y_{12}| \leq |y_{23}|$, the realization is accomplished by

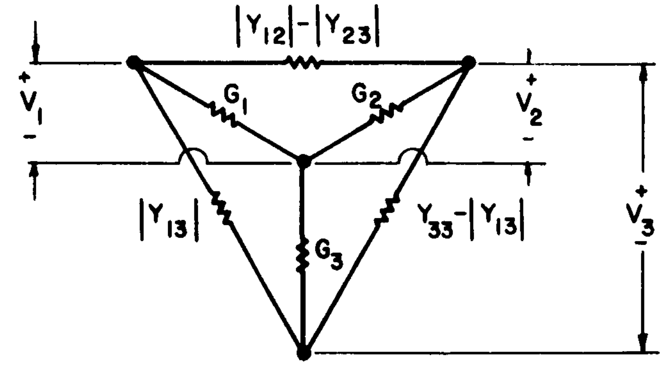
$$A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4-44}$$

The corresponding networks are shown in Figures 22, 23, and 24.



$$G_1 = y_{11} - |y_{13}| - |y_{12}|$$
$$G_2 = y_{22} - |y_{23}| - |y_{12}|$$
$$G_3 = y_{33} - |y_{13}| - |y_{23}|$$

Fig. 22 Realization of 3-port for $y_{12} \geq 0$

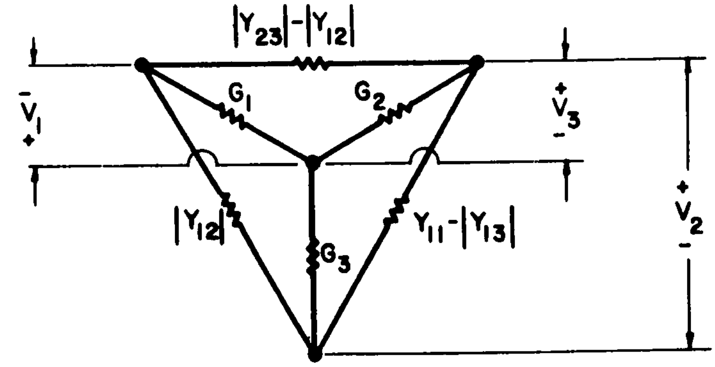


$$G_1 = y_{22} - |y_{12}|$$

$$G_2 = y_{11} + |y_{23}| - |y_{12}| - |y_{13}|$$

$$G_3 = |y_{13}| - |y_{23}|$$

Fig. 23 Realization of 3-port for $y_{12} < 0$ and $|y_{23}| \leq |y_{12}|$



$$G_1 = y_{22} - |y_{23}|$$

$$G_2 = y_{33} + |y_{12}| - |y_{23}| - |y_{13}|$$

$$G_3 = |y_{13}| - |y_{12}|$$

Fig. 24 Realization of 3 port for $y_{12} < 0$ and $|y_{23}| \geq |y_{12}|$

Corollary 1 Dominance is a sufficient condition for the realization of any symmetric 3 x 3, as a short-circuit admittance matrix, on four terminals.

Proof If $y_{12} \geq 0$, this is obvious. If $y_{12} < 0$, dominance insures that

$$y_{11} \geq | y_{13} | + | y_{12} | \tag{4-45}$$

and

$$y_{33} \geq | y_{23} | + | y_{13} | \tag{4-46}$$

Thus, the inequalities of the previous theorem are certainly satisfied by dominance.

Notice, however, that the inequalities of Theorem 20 show that dominance is certainly not necessary.

Corollary 2 Paramountcy is not a sufficient condition for realization.

Proof Consider

$$Y = \begin{bmatrix} 6 & 1 & +3 \\ 1 & 7 & -2 \\ +3 & -2 & 4 \end{bmatrix} \tag{4-47}$$

Since $y_{12} > 0$ in the standard form, dominance is necessary for realization. Since Y is not dominant it is not realizable; however, Y is certainly paramount. Hence, five terminals are necessary for therealization of a paramount matrix[24].

Since dominance is not necessary and paramountcy is not sufficient, it is apparent that the no-gain property of resistor n-ports is not sufficient to characterize the fact that there are no transformers.

An application of the previous theorem is given by the following example.

$$Z = \begin{bmatrix} \frac{9}{55} & -\frac{4}{55} & -\frac{3}{110} \\ -\frac{4}{55} & \frac{14}{55} & -\frac{17}{110} \\ -\frac{3}{110} & \frac{17}{110} & \frac{14}{55} \end{bmatrix} \tag{4-48}$$

inversion yields

$$Y = \begin{bmatrix} 9 & 5 & 4 \\ 5 & 9 & 6 \\ 4 & 6 & 8 \end{bmatrix} \tag{4-49}$$

conversion to standard form yields

$$Y = \begin{bmatrix} 9 & -5 & +6 \\ -5 & 9 & -4 \\ +6 & -4 & 8 \end{bmatrix} \tag{4-50}$$

The inequalities insuring realizability are certainly satisfied by this example. However, the conversion to standard form involved interchanging the first and second rows and columns and multiplying the second row and second column by -1. The network realizing the original Z is shown in Fig. 25.

$$Z = \begin{bmatrix} \frac{9}{55} & -\frac{4}{55} & -\frac{3}{110} \\ -\frac{4}{55} & \frac{14}{55} & -\frac{17}{110} \\ -\frac{3}{110} & -\frac{17}{110} & \frac{14}{55} \end{bmatrix}$$

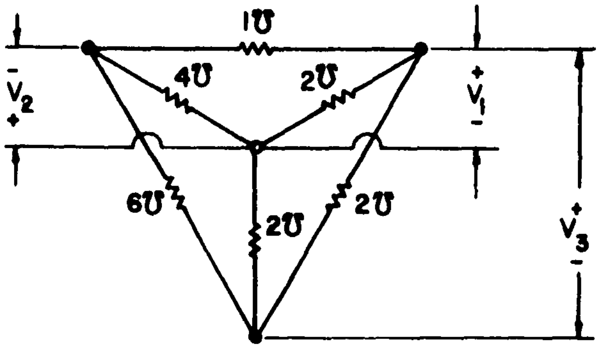


Fig. 25 Synthesis of an open-circuit imedpance

APPENDIX I

THE CUT-SET MATRIX

The classical definition of a cut-set matrix is given, and it is shown to be equivalent to the definition given in the first section. In this context, it is assumed that an arc does not contain its endpoint; thus the removal of an arc does not remove a node. An isolated node will also be considered a connected graph.

Definition 20 (22) A cut-set of a connected graph is a set of arcs such that the removal of these arcs yields a disconnected graph, and no subset of these arcs have the same property. The removal of a cut-set clearly yields two disjoint connected subgraphs. Now since the graph is connected, it has a tree. Let the arcs of the graph which are not in the tree be called links. Clearly any branch of a tree is a cut-set of the tree. Thus a cut-set of the graph is a branch from a tree together with some of the links of the tree. Furthermore, the links that are included are characterized by the fact that they form a circuit with the tree branch under consideration. Hence for a connected graph of n + 1 nodes, any tree of the graph defines a set of n distinct cut-sets; these cut-sets are called a fundamental collection of cut-sets with respect to this tree. It is also convenient to assign a direction to these cut-sets. The direction of the cut-set will be taken as the direction of the tree branch, and a link is said to agree with the direction of the cut-set if it is directed oppositely to thetree branch in the circuit which it forms with the tree branch. Otherwise the link is said to disagree with the direction of the cut-set. The fundamental cut-set matrix is defined as follows.

Definition 21 (22) The fundamental cut-set matrix of a connected graph of n+1 nodes and b arcs is a b x n matrix C_f such that

$$C_f = \left[c_{ij}\right]$$

where

$$c_{ij} = +1 \text{ if arc i is contained in cut-set j and agrees in direction,}$$
$$= -1 \text{ if arc i is contained in cut-set j and disagrees in direction.}$$

and $\qquad = 0$ otherwise,

where the cut-sets are taken as the n fundamental cut-sets of some tree of the graph.

Now consider the elements in the product SB, which will be denoted by Q. First consider the rows of S which have exactly two non-zero elements s_{ki} and s_{kj}; these elements are of opposite sign. Thus

$$q_{kp} = s_{ki}\, b_{ip} + s_{kj}\, b_{jp} \tag{I-1}$$

Now if b_{ip} and b_{jp} are both non-zero, then c_{kp} is zero since the elements in any column of B have the same sign. Thus if arc k is not incident at the reference and

if i and j are the endpoints of this arc, then q_{kp} is zero if the ground paths from nodes i and j both include branch p. Clearly, link k does not form a circuit with branch p if the ground paths from both of the nodes of p include k. On the other hand, if one of the terms b_{ip} or b_{ip} is zero and the other non-zero, then link k does form a circuit with branch p, and q_{kp} is +1 or -1 in accordance as p and k are directed oppositely or in the same sense in this circuit respectively. Furthermore, if the rows of S with only one non-zero element are considered, then a similiar result is obtained. Thus,

$$Q = C_f \tag{I-2}$$

APPENDIX II

THE CEDERBAUM ALGORITHM

Even in the case when there are zeros in the Y matrix, the minimum magnitude non-zero term has a special significance.

Theorem 21 If Y is a realizable admittance matrix and the minimum magnitude off diagonal term corresponds to ports i and j and has the value g, then there is a unique link whose corresponding conductance is g such that this link is the only link common to the cut-sets defined by i and j.

Proof Consider the longest tree path which includes i and j and whose initial and terminal nodes are connected by a non-zero conductance of value z; see Fig. 26. Let the corresponding initial and terminal branches of this path be p and r.

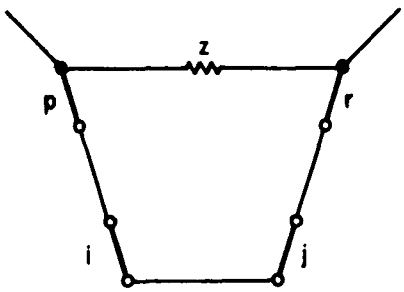


Fig. 26 Tree path for minimum magnitude ports

If all ports but p and r are shorted, then

$$| y_{pr} | = y \qquad , \tag{II-1}$$

since there is no path of greater length which has a link connecting the initial and terminal nodes. Clearly

$$| y_{ij} | \geq y \tag{II-2}$$

However, $|y_{ij}|$ is a minimum; hence

$$y = g \tag{II-3}$$

Now any link which is contained in both the cut-sets defined by i and the cut-sets defined by j must form a circuit with some path through i and j. Assume there is another link of conductance x with this property. Then

$$| y_{ij} | \geq x + g \qquad , \tag{II-4}$$

which is certainly impossible. Thus y is the only link with this property. Consider the determination of all cut-sets which contain link y; these are all the ports contained in the tree path from p to r. Let i and j be oriented such that y_{ij} is positive. Then the

only ports contained in this path are ports s such that

$$y_{is} > 0 \qquad \text{(II-5)}$$

and

$$y_{js} > 0$$

The proof of this is immediate. Any port s with this property must form a third order linear tree with i and j when all other ports are shorted. If s is not contained in the path from p to r, then

$$|y_{sj}| < |y_{ij}| \qquad \text{(II-6)}$$

This is impossible unless $|y_{sj}|$ is zero.

If a given Y matrix is assumed to be realizable then this result may be used to determine the value of one of the link conductances and uniquely determine an entire row of the corresponding cut-set matrix. Let $\underset{\sim}{a}$ denote the row that is obtained. Then

$$Y - \underset{\sim}{a}' \, g \underset{\sim}{a} \qquad \text{(II-7)}$$

is the admittance matrix of an identical structure except that conductance g has been removed. By repeating this process, one obtains an admittance matrix whose off diagonal terms are all zero. Since there are only zeros in the off-diagonal positions, there are no links. Thus the remaining rows of the cut-set matrix correspond to the tree, and after a suitable permutation, the corresponding cut-set matrix is the unit matrix. Hence the conductances corresponding to thetree branches are just the diagonal terms. If this procedure is applied to an arbitrary matrix Y, then it can be factored into the for C'GC where the matrix C is the corresponding cut-set matrix if and only if Y is realizable. In order to obtain a complete synthesis, the matrix C must be tested to determine if it is a realizable cut-set matrix. The algorithm may be summarixed as follows: if a matrix Y is a realizable admittance matrix then it has a unique factorization into C'DC where C is a cut-set matrix and D a diagonal matrix of positive constants. It follows from thisresult that the realization of a cut-set matrix is equivalent to the synthesis of a resistor n-port on n+1 nodes. Indeed suppose the matrix Q'Q is a realizable admittance matrix on a network of unit conductances. Then there exists a cut-set matrix C such that

$$Q'Q = C'C \qquad \text{(II-8)}$$

From the factorization theorem, it is evident that if C is a cut-set matrix then

$$C = Q \qquad \text{(II-9)}$$

Thus if

$$C \neq Q, \qquad \text{(II-10)}$$

then C is not a cut-set matrix.

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